Abstract—This paper introduces a two-user discrete-time input-dependent Gaussian noise optical multiple access channel (OMAC) which is applicable to a number of optical wireless links, most notably space optical communications as well as visible light communications. Under nonnegativity and peak intensity constraints, it is shown that generating the code-books of both users according to discrete distributions with finite supports achieves the largest sum-rate in the network. In other words, sum-capacity-achieving distributions for this channel are discrete with a finite number of mass points.

I. INTRODUCTION

Optical wireless communications is utilized in space optical communications and visible light communications [1], [2]. It is often based on intensity modulation and direct detection (IM-DD), where the channel input modulates the intensity of the emitted light. Thus, the input signal is proportional to the light intensity and is nonnegative. The receiver is usually equipped with a photodetector which absorbs integer number of photons and generates a real valued output corrupted by noise.

Based on the distribution of this corrupting noise, there are several channel models for the underlying optical wireless channels. Free space optical (FSO) channels [1], [3], optical channels with input-dependent Gaussian noise [1], [4] and Poisson optical channels [1], [5]–[7] are the most widely used channel models for optical wireless communications. Among the existing channel models for optical wireless communications, the most accurate model that can capture most of the optical channel impairments, is the Poisson channel model. Nevertheless, when the ambient light in the environment is dominant, the noise in the optical channel can be well-approximated by a Gaussian distribution whose variance may or may not depend on the channel input [1], [4].

Studying the communications performance limits (such as the channel capacity) of these channel models from an information-theoretic point of view is rather difficult. This is because the channel input must satisfy nonnegativity, peak and average intensity constraints due to eye safety and practical considerations [1]. Considering the single-user capacity of the aforementioned channel models, the works in [6], [8] show that the capacity-achieving input distributions are discrete with a finite number of mass points when the channel input is constrained by nonnegativity, peak and average intensity constraints. This is on the contrary to the case of the Gaussian channels with average power constrained inputs, where Gaussian input distribution with an infinite support is capacity-achieving [9].

Information-theoretic studies have also been performed for the multiuser versions of FSO and Poisson channel models [10]–[13]. For instance, the work in [10] considers the Gaussian multiple access channel with amplitude constraints and establishes that the sum-capacity-achieving distributions are discrete with a finite support. These results are directly applicable to the free space optical multiple access channel (FSO-MAC) with nonnegativity and peak intensity constraints. Furthermore, [11] provides tight bounds on the capacity region of FSO-MAC across several intensity regimes (low, moderate, and high). For a continuous-time Poisson OMAC, Lapidoth et al. establish the capacity region of the Poisson MAC for the two-user case. The authors show that for achieving every point on the boundary of the capacity region, the input distributions for both users should be binary with an infinite transmission bandwidth. The discrete-time Poisson OMAC has also been considered in [13], where authors study the two-user case and prove that sum-capacity-achieving input distributions of both users must be discrete with a finite number of mass points when peak intensity constraint is considered.

In this work, we study a discrete-time input-dependent Gaussian noise OMAC which consists of two optical transmitters and a receiver. In this setup, the input signals are restricted by nonnegativity, peak and average intensity constraints. Using an IM-DD system, the photodetector at the receiver counts.
Based on (3), the channel output $Y_i$ decode over constraints (1)–(2). The receiver collect the received symbols $i \in \{1,2\}$ to a codeword of length $n$ according to a zero-mean Gaussian distribution with variance $\eta^2$ and given by [4] 

$$R\Delta \equiv \sup_{F_{X_1} \in \mathcal{F}_1, F_{X_2} \in \mathcal{F}_2} I(X_1, X_2; Y).$$

where $I(X_1, X_2; Y)$ is the mutual information between the channel inputs $X_1$ and $X_2$ and the channel output $Y$. $F_{X_i}, i \in \{1,2\}$ is the distribution function of $X_i$, and $\mathcal{F}_i$ denotes the set of distribution functions defined as 

$$\mathcal{F}_i \triangleq \left\{ F_X : \int_0^{A_i} dF_X(x) = 1, \int_0^{A_i} x dF_X(x) \leq E_i, \quad i \in \{1,2\} \right\}.$$

III. SUM-CAPACITY-ACHIEVING DISTRIBUTIONS

This section presents the main results of the paper related to characterization of the input distributions that attains the sum-capacity of the input-dependent Gaussian noise OMAC with nonnegativity, peak and average intensity constraints. These results are summarized in the following theorems.

**Theorem 1.** Let $X_1$ and $X_2$ be two independent random variables. For any $F_{X_i} \in \mathcal{F}_i$, the optimal solution to the following optimization problem 

$$\sup_{F_{X_i} \in \mathcal{F}_i} I(X_1, X_2; Y)$$

is unique and is discrete with a finite number of mass points.

**Proof.** For convenience, the proof is presented in Section IV.

To prove Theorem 1, we first show that the set of input distributions $\mathcal{F}_i$, $i \in \{1,2\}$ satisfying (6) is compact and convex. We then show that the objective functional in (7) is continuous, strictly concave, and weakly differentiable in the input distribution $F_{X_i} \in \mathcal{F}_i$. Hence, we conclude that the solution to the optimization problem (7) exists and is unique. We continue the proof by deriving the necessary and sufficient conditions (also known as Karush-Kuhn-Tucker conditions, or KKT) for the optimality of the input distribution $F_{X_i}$ and finally, by means of contradiction we show that the optimal distribution is discrete and possesses a finite number of mass points.

A direct consequence of Theorem 1 is given by the following corollary.

**Corollary 1.** For a two-user discrete-time input-dependent Gaussian noise OMAC with nonnegativity, peak and average intensity constraints, the sum-capacity-achieving input distributions are discrete with a finite number of mass points.

IV. PROOF OF THEOREM 1

In this section, we present the proof of Theorem 1. To this end, we first review some preliminaries that are used throughout the proof. Then, we present a few lemmas that are used to establish the theorem.
where \[ \text{following bounds } \{15\} \]

It is straightforward to show that

\[ P(y; F_{X_1}, F_{X_2}) = \int_0^{\mathcal{A}_1} \int_0^{\mathcal{A}_2} p(y|x_1, x_2) dF_{X_2} dF_{X_1}, \quad y \in \mathbb{R}, \]

where \( p(y|x_1, x_2) \) is given by (3). Based on (3), we have the following bounds [15]

\[ k_1 e^{-k_3(y-x_1-x_2)^2} \leq p(y|x_1, x_2) \leq k_3 e^{-k_4(y-x_1-x_2)^2}, \quad (9) \]

for some positive finite values \( k_1, k_2, k_3, k_4, \) where

\[ k_1 = \frac{1}{\sqrt{2\pi\sigma^2(\mathcal{A}_1, \mathcal{A}_2)}}, \quad k_2 = \frac{1}{2\sigma^2}, \quad (10) \]

\[ k_3 = \frac{1}{2\sigma^2}, \quad k_4 = \frac{1}{2\sigma^2(\mathcal{A}_1, \mathcal{A}_2)}. \quad (11) \]

Next, we define \( \gamma(y) \) and \( \Gamma(y) \) as follows

\[ \gamma(y) = \min_{0 \leq x_1 \leq \mathcal{A}_1, 0 \leq x_2 \leq \mathcal{A}_2} k_1 e^{-k_3(y-x_1-x_2)^2}, \quad (12) \]

\[ \Gamma(y) = \max_{0 \leq x_1 \leq \mathcal{A}_1, 0 \leq x_2 \leq \mathcal{A}_2} k_3 e^{-k_4(y-x_1-x_2)^2}. \quad (13) \]

It is straightforward to show that

\[ \gamma(y) \leq P_Y(y; F_{X_1}, F_{X_2}) \leq \Gamma(y). \quad (14) \]

We now define the sum-rate density \( \Lambda(x_1; F_{X_1}) \) as

\[ \Lambda(x_1; F_{X_1}) = -\frac{1}{2} \mathbb{E}_{X_1} \left[ \log \left( 2\pi e \sigma^2(x_1, x_2) \right) \right]. \quad (17) \]

where \( \psi(y) \) is defined as \( \log \left( P_Y(y; F_{X_1}, F_{X_2}) \right) \). It is easy to observe that

\[ I(X_1, X_2; Y) = \int_0^{\mathcal{A}_1} \Lambda(x_1; F_{X_1}) dF_{X_1}. \quad (18) \]

In the sequel, we assume, without loss of generality, that \( F_{X_1} \) is fixed. Consequently, one can define the mapping \( \Xi : \mathcal{F}_1 \rightarrow \mathbb{R}^+ \) such that \( \Xi(F_{X_1}) = I(X_1, X_2; Y) \). Now, we are ready to present the proof of Theorem 1.

### B. Proof of Theorem 1

For convenience, the proof is streamlined into a few lemmas which we state below.

**Lemma 1.** The feasible set \( \mathcal{F}_1 \) is convex and sequentially compact in the Levy metric sense.

**Proof.** The proof follows along similar lines as [16, Appendix A.1].

**Lemma 2.** The functional \( \Xi(F_{X_1}) = I(X_1, X_2; Y) \) is continuous in \( F_{X_1} \).

**Proof.** The proof uses the bounds in (9) and (16), and follows along similar lines of [17, Section IV].

From Lemma 1 and Lemma 2, \( \Xi(F_{X_1}) \) is continuous in \( F_{X_1} \) over \( \mathcal{F}_1 \) which itself is a compact and convex set, then by Extreme Value theorem, \( \Xi(F_{X_1}) \) is bounded above and attains its supremum. That is, the supremum in (7) is achievable by at least one input distribution \( F_{X_1} \).

**Lemma 3.** The functional \( \Xi(F_{X_1}) \) is strictly concave in \( F_{X_1} \).

**Proof.** The proof follows along similar lines of [13, Proposition 6].

**Lemma 4.** Defining \( F_{X_1,\psi} = (1 - \theta)F^*_{X_1} + \theta F_{X_1} \), \( \forall F^*_{X_1}, F_{X_1} \in \mathcal{F}_1, \theta \in [0, 1] \), the weak derivative of \( \Xi(F_{X_1}) \) at \( F^*_{X_1} \) denoted by \( D(\Xi(F^*_{X_1})) \) exists and is equal to

\[ D(\Xi(F^*_{X_1})) = \lim_{\theta \to 0} \frac{\Xi((1 - \theta)F^*_{X_1} + \theta F_{X_1}) - \Xi(F^*_{X_1})}{\theta} = \int_0^{\mathcal{A}_1} \Lambda(x_1; F^*_{X_1}) dF_{X_1} - \Xi(F^*_{X_1}). \quad (19) \]

**Proof.** The proof is based on the definition of the weak derivative and follows along similar lines of [17, Section IV] and is omitted for brevity.

Now, from Lemma 1, Lemma 3, and Lemma 4, we have a strictly concave and weak-differentiable functional \( \Xi(F_{X_1}) \) over \( \mathcal{F}_1 \) which is a convex set, then the necessary and sufficient conditions for an input distribution \( F^*_{X_1} \) to be optimal is

\[ D(\Xi(F^*_{X_1})) \leq 0. \quad (20) \]

Furthermore, the mapping defined as \( g(F_{X_1}) = \int_0^{\mathcal{A}_1} x dF_{X_1} - \lambda_1 \mathbb{E}[X_1] \) from \( \mathcal{F}_1 \rightarrow \mathbb{R} \), where \( \lambda_1 > 0 \), is continuous in \( F_{X_1} \), strictly concave, and weakly differentiable. As a result, following along the similar steps of [14, Corollary 1] yields the following necessary and sufficient conditions for the optimality of \( F^*_{X_1} \)

\[ \Lambda(x_1; F^*_{X_1}) - \lambda_1 x_1 \leq \Xi(F^*_{X_1}) - \lambda_1 \mathbb{E}[X_1], \quad \forall x_1 \in [0, \mathcal{A}_1], \quad (21) \]

\[ \Lambda(x_1; F_{X_1}) - \lambda_1 x_1 = \Xi(F^*_{X_1}) - \lambda_1 \mathbb{E}[X_1], \quad \forall x_1 \in S_{F^*_{X_1}}, \quad (22) \]

\[ \lambda_1 (\mathbb{E}[X_1] - E_1) = 0. \quad (23) \]

where \( S_{F^*_{X_1}} \subset [0, \mathcal{A}_1] \) is the support set of \( F^*_{X_1} \). We are now ready to establish that the optimal input distribution \( F^*_{X_1} \) is...
discrete with a finite number of mass points. To prove the discreteness, we resort to a contradiction argument using the KKT conditions in (21)–(23). To this end, we first present a discrete with a finite number of mass points. To prove the

Lemma 5. The extension of the function \( \Lambda(x_1; F_{X_1}) - \lambda x_1 \) to the open connected set \( O \) is analytic, where \( \Re \{w\} \) is the real part of the complex variable \( w \).

Proof. The proof follows along similar lines of [17, Section IV].

Next, we assume that \( S_{F_{X_1}} \) has an infinite number of elements. In view of the optimality condition (22), the analyticity of \( \Lambda(w; F_{X_1}) - \lambda_1 w \) over \( O \), and the Identity Theorem from complex analysis along with the Bolzano-Weierstrass Theorem, if \( S_{F_{X_1}} \) has an infinite support, we can deduce that \( \Lambda(w; F_{X_1}) - \lambda_1 w = \Re(F_{X_1}) - \lambda_1 \mathcal{E}_1, \forall w \in O \). Since \((-1/\eta^2, +\infty) \subset O\), we have

\[
\Lambda(x_1; F_{X_1}) - \lambda_1 x_1 = \Re(F_{X_1}) - \lambda_1 \mathcal{E}_1, \forall x_1 \in (-1/\eta^2, +\infty).
\]

Expanding \( \Lambda(x_1; F_{X_1}) \) based on (17), we can rewrite (24) as

\[
\int_{\Re} \int_{0}^{\mathcal{A}_2} p(y|x_1, x_2) \log \left[ \frac{1}{F_Y(y; F_{X_1}, F_{X_2})} \right] dF_{X_2} dy \\
= \frac{1}{2} \mathbb{E}_{X_1} \left[ \log \left( 2\pi e\sigma^2(x_1, x_2) \right) \right] + \Re(F_{X_1}^*) + \lambda_1 (x_1 - \mathcal{E}_1),
\]

for all \( x_1 \in (-1/\eta^2, +\infty) \). Denoting the left hand side of (25) by \( T(x_1; F_{X_1}^*) \) and using the bounds in (16), we lower bound \( T(x_1; F_{X_1}^*) \) as

\[
T(x_1; F_{X_1}^*) \geq \int_{\Re} \int_{0}^{\mathcal{A}_2} p(y|x_1, x_2) \log (1/\Gamma(y)) dF_{X_2} dy \\
= \int_{\Re} \int_{0}^{\mathcal{A}_2} p(y|x_1, x_2) \log (1/k_1^3) + k_4 y^2 dF_{X_2} dy \\
+ \int_{\Re} \int_{0}^{\mathcal{A}_2} p(y|x_1, x_2) \log (1/k_3^0) dF_{X_2} dy \\
+ \int_{\Re} \int_{0}^{+\infty} p(y|x_1, x_2) \log (1/k_3) dF_{X_2} dy \\
+ k_4 (y - \mathcal{A})^2 dF_{X_2} \\
= c_1 + \int_{\Re} \int_{0}^{\mathcal{A}_2} k_4 y^2 p(y|x_1, x_2) dF_{X_2} dy \\
+ \int_{\Re} \int_{0}^{+\infty} k_4 (y - \mathcal{A})^2 p(y|x_1, x_2) dF_{X_2} dy \\
= c_1 + \int_{\Re} \int_{0}^{\mathcal{A}_2} k_4 y^2 p(y|x_1, x_2) dF_{X_2} dy \\
- \int_{0}^{\mathcal{A}_2} k_4 y^2 p(y|x_1, x_2) dF_{X_2} \\
x \leq k_4 \mathcal{A}^2 k_3.
\]

The extension of the function \( \Lambda(x_1; F_{X_1}) - \lambda x_1 \) to some open connected set in the complex plane \( \mathbb{C} \) is analytic. Therefore, generating the codebooks of both users according to discrete input distributions with a finite number of mass points achieves the sum-capacity of the input-dependent Gaussian noise OMAC.

V. NUMERICAL RESULTS

This section provides numerical inspections for the sum-capacity of the input-dependent Gaussian noise OMAC with nonnegativity, peak, and average intensity constraints.

Figure 1 and Figure 2 illustrate the sum-rate density for the optimal input distributions \( F_{X_1}^* \) and \( F_{X_2}^* \) for \( \mathcal{A}_1 = \mathcal{A}_2 = 5, \sigma_0^2 = 1, \) and \( \eta^2 = 0.25 \). We numerically found that for these parameters, the optimal input distribution for \( X_1 \) is ternary with mass points located at \( x = 0.26986, 5 \) with probability masses \( 0.7159, 0.1826, 0.1015 \), respectively with the corresponding Lagrangian multiplier \( \lambda_1 = 0.1627 \). Moreover, the optimal input distribution for \( X_2 \) is binary with mass points at \( 0.5 \) with probability masses \( 0.8, 0.2 \), respectively with the corresponding Lagrangian multiplier \( \lambda_2 = 0.155 \). We observe that \( \Re(F_{X_1}^*) - \Lambda(x_1; F_{X_1}^*) + \lambda (x_1 - \mathcal{E}_1), i \in \{1, 2\} \) is generally nonnegative and is equal to zero at the optimal mass points; verifying the optimality conditions in (21)–(23).

Figure 3 plots the sum-capacity of the input-dependent Gaussian noise OMAC with nonnegativity, peak, and average intensity constraints versus \( \mathcal{A}_1 \). In the simulations, we choose \( \mathcal{A}_2 = \mathcal{A}_1^2, \mathcal{E}_1 = \mathcal{A}_1^4, \) and \( \mathcal{E}_2 = \mathcal{A}_1^2 \) for \( \sigma_0^2 = 1 \) and \( \eta^2 = 0.25 \). We observe that the sum-capacity increases as the allowed
Fig. 2. Illustration of $\Xi(F_{x_1}^*) - \Lambda(x_1; F_{x_1}^*) + \lambda (x_1 - \epsilon_1)$ yielded by the optimal input distribution when $\sigma_0^2 = 1$, $\eta^2 = 0.25$, $\mathcal{A}_1 = \mathcal{A}_2 = 5$, and $\epsilon_1 = \epsilon_2 = 1$. peak intensity constraints increases. Additionally, in the figure, we plot the sum-rate of the network achieved by binary input distributions for both $X_1$ and $X_2$. As can be seen from the figure, with the chosen parameters, binary input distributions achieve the sum-capacity when $\mathcal{A}_1 \leq 7$ and is negligibly inferior to the sum-capacity for $7 \leq \mathcal{A}_1 \leq 10$.

VI. CONCLUSIONS

We studied the discrete-time input-dependent Gaussian noise OMAC with nonnegativity, peak, and average intensity constraints. We formally characterized the sum-capacity achieving input distributions to be unique and discrete with a finite number of mass points. Our numerical results implied that generating the codebooks of both users according to binary input distributions with mass points located at the origin and the value of the peak intensity constraint can achieve the sum-capacity of the network for small values of the peak intensity constraints. As a future work, we plan to extend the optimality of the discrete input distributions for achieving the entire boundary of the capacity region of the input-dependent Gaussian noise OMAC.

APPENDIX A

AN UPPER BOUND FOR $f(x_1, x_2)$

We start by noting that $p(y|x_1, x_2) \leq k_3 e^{-k_4(y-x_1-x_2)^2}$ due to (9), hence we get

$$\int_{\mathcal{A}} y p(y|x_1, x_2) \, dy \leq \int_{\mathcal{A}} y k_3 e^{-k_4(y-x_1-x_2)^2} \, dy$$

$$= (x_1 + x_2) k_3 \int_{\mathcal{A} + x_1 + x_2} e^{-k_4 y^2} \, dy$$

$$+ k_3 \int_{\mathcal{A} + x_1 + x_2} y e^{-k_4 y^2} \, dy$$

$$\leq k_3 \frac{e^{-k_4(\mathcal{A} + x_1 + x_2)^2}}{2k_4} \sqrt{\frac{\pi k_4}{4k_2}} (x_1 + x_2)$$

$$\times \left[ 1 - \text{erf}(\sqrt{k_4(\mathcal{A} + x_1 + x_2)}) \right]$$

$$\leq \alpha (1 + x_1 + x_2),$$

(27)

where $\text{erf}(x) \overset{\Delta}{=} \int_0^x e^{-t^2} \, dt$ is the error function, and $\alpha \overset{\Delta}{=} \max \left\{ \sqrt{\frac{\pi k_1}{3k_2}}, \sqrt{\frac{\pi k_1}{3k_2}} \right\}$ is a positive constant.

REFERENCES


