On the Error Rates of Space-time Codes for Millimeter-wave Communication

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Abstract

The Nakagami-\(m\) distribution captures better the multitude of fading scenarios experienced by millimeter-wave (mmWave) signals than the Rayleigh and Rician distributions. Moreover, mmWave signals are particularly sensitive to blockage. But, the widely accepted design criteria for space-time codes (STCs) were developed for the Rayleigh and Rician fading channels and underestimate the impact of blockage on the error rates. STCs designed based on such guidelines achieve suboptimal diversity and coding gains over the Nakagami-\(m\) fading channel. Hence, we derive tight upper and lower bounds on the diversity and coding gains of any STC over the Nakagami-\(m\) fading channel. We then identify necessary and sufficient properties of STCs that achieve the upper bounds, which leads to a fundamental trade-off between maximizing the diversity and coding gains, i.e., achieving the upper bound for one automatically leads to achieving the lower bound on the other. We also investigate the effect of blockage on the error rates using stochastic geometry. We show that blockage reduces the coding gain but not the diversity gain. Numerical simulations show that the STCs that satisfy that derived conditions achieve full diversity, whereas codes that are otherwise optimal for Rayleigh and Rician channels do not. Furthermore, in a typical indoor environment, blockage reduces the coding gain by 1.5 dB for BERs less than 10\(^{-3}\).

Keywords

MIMO, mmWave, space-time coding, Nakagami-\(m\), diversity gain, coding gain, blockage, 5G

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I. INTRODUCTION

Communicating via the fifth- and sixth-generation standards is anticipated to occur at wireless millimeter-wave (mmWave) frequencies (30 GHz – 300 GHz), in addition to the 700 MHz – 2.6 GHz bandwidth used in previous and current standards [2]–[4]. mmWave communication promises alleviating spectrum scarcity, providing high data rates to users, and improving reliability [5]. One of the main problems encountered when designing communication systems for mmWave communication is the design of multi-input multi-output (MIMO) techniques for mmWave communication. This is especially interesting because established signal processing and coding schemes for the radio frequency channel assume certain characteristics about the channel that are no longer valid for the mmWave channel. In this paper, we focus on optimizing space-time codes (STCs) for mmWave communication.

To achieve high reliability, STCs can be used [1]. STCs involve coding simultaneously over time and multiple antennas to increase reliability. To apply STCs for mmWave communication, the peculiarities of the mmWave channel should be considered in the design and analysis of STCs. For instance, due to the high frequency and, hence, highly directive nature of mmWave signals, mmWave communication is likely to experience various fading scenarios [6]. Deep fades are expected to occur frequently when there is no line-of-sight (LOS) link between the transceivers, whereas favorable fading conditions will likely happen when the LOS link is not interrupted. This phenomenon does not exist in conventional radio frequency (RF) channels where a large number of multipath components arrive at the receive antennas from many different directions, and, hence, the Rayleigh distribution adequately models fading.

But, for mmWave channels, the Nakagami-\(m\) distribution provides a more generalized fading distribution that captures instances of deep and favorable fades via varying the parameter \(m\), while maintaining mathematical tractability. Hence, it has been found increasingly helpful in the modeling and analysis of performance metrics for mmWave communications, including: coverage probability, outage probability, and error probability [7]–[14]. One of the early instances of using the Nakagami-\(m\) distribution to model mmWave channels is [7], where the authors use the Nakagami-\(m\) distribution to model LOS and NLOS links in their proposal of a stochastic geometry based framework for analyzing coverage and rate for mmWave cellular networks. More recent studies expand upon this model to study different performance metrics and signal
processing methods for mmWave communication. In [10], the bit error probability (BEP) of a multihop relaying system is studied for communication over mmWave bands. The proposed system is intended to reduce the high outage probability over mmWave channels caused by the sensitivity of mmWave signals to blockage. Therein, the Nakagami-\(m\) distribution is found to provide a flexible yet tractable model of multipath fading. Working independently, [14] derived coverage and BEP results for a multihop relaying system intended for mmWave communication. The Nakagami-\(m\) distribution is similarly assumed for modeling different fading severity over LOS and NLOS links. Utilizing the Nakagami-\(m\) distribution as an accurate and convenient model for multipath fading over mmWave channels, and a \((\min,+)-\)algebra, the delay and capacity of a traffic dispersion scheme are studied in [11]. In another recent study [15], the outage probability in a device-to-device (D2D) system communicating over mmWave frequencies is derived, where the Nakagami-\(m\) distribution models the LOS and NLOS fading distributions.

Undoubtedly, the aforementioned studies further facilitate mmWave communication by suggesting methods that improve outage due to the low penetration nature of mmWave signals. Nonetheless, the potential for improved reliability over the mmWave channel by coding over the multiple antennas at the transceiver has not been adequately studied. In this paper, we study the error rates of any STC over the mmWave channel to propose design guidelines for such codes to achieve the optimal diversity and coding gains, and, hence, increase reliability.

Though the error performance analysis of STCs is a relatively mature topic [16]–[23], closed-form expressions for the diversity and coding gains over MIMO Nakagami-\(m\) fading channels have not been derived. In the seminal work of Tarokh \textit{et al.} [21], design criteria for MIMO Rician and Rayleigh fading channels were proposed from derivations of the average pairwise error probability (PEP). But, these results for the Rayleigh and Rician MIMO channels do not apply for Nakagami-\(m\) channel. Furthermore, although the PEP for single-input-single-output (SISO) and single-input-multiple-output (SIMO) systems under Nakagami-\(m\) fading channels is known and can be easily derived, analyzing the PEP over MIMO Nakagami-\(m\) fading channels is more involved. Moreover, the results for the Nakagami-\(m\) SISO and SIMO channels do not generally apply for the Nakagami-\(m\) MIMO channel. This calls upon studying the MIMO case separately. One of the challenges, as will be discussed in detail in Subsection III-B, is the lack of a tractable, closed-form expression for the exact distribution of the envelope of the sum of complex-valued random variables with Nakagami-\(m\)-distributed amplitudes and uniformly
distributed phases. To circumvent this, several studies [19], [20] propose expressions for the PEP over Nakagami-$m$ fading MIMO channels in terms of an infinite series and integral forms, respectively. These integral form expressions, however, preclude the deduction of diversity and coding gains and inference of design criteria as performed in [21] for Rayleigh and Rician fading channels. In [24], [25], the diversity gain for the Nakagami-$m$ fading channel is studied for orthogonal space-time-block codes (OSTBCs) and for maximum ratio combining (MRC), respectively. This paper analyzes the diversity and coding gains from a closed-form tight upper bound on the PEP for any STC over the Nakagami-$m$ fading channel. Due the recent interest in modeling multipath fading in mmWave channels via the Nakagami-$m$ distribution [6], [8], [12], [13], the design guidelines presented herein are important improving reliability via STCs for mmWave communication.

In contrast, classical code design criteria for Rayleigh and Rician fading may lead to sub-optimal diversity and coding gains over the Nakagami-$m$ fading channel. For instance, well-known codes such as perfect STCs including its subcode the Golden Code (GC) [26] have been designed and optimized under the assumption that fading is Rayleigh distributed. As mentioned, this assumption may no longer hold under for the mmWave channel. In fact, as discussed, the Nakagami-$m$ distribution provides a more accurate model for fading for the mmWave channel. Such STCs that are diversity optimal over the Rayleigh fading channel cannot achieve the maximum achievable diversity gain over the Nakagami-$m$ fading channel. We indeed show via analysis and numerical simulations that this true. Moreover, we derive design guidelines for STCs operating over the mmWave channel to achieve optimal error performance.

Further, the mmWave channel has been shown to be highly directive through measurement campaigns [5], [28]. Consequently, blockage is expected to be a limiting factor in the performance of such channels. As a results, the effect of blockage on the performance of the channel has been of interest to the community [2], [5], [7], [8], [13], [29]–[33]. In [29], a framework for analyzing the performance of mmWave cellular networks using stochastic geometry is proposed. In [8] and [7], objects that can cause blockage are modeled using a Poisson point process (PPP) with a certain density that is considered to be a system parameter. From stochastic geometry, the Thinning Theorem can then be invoked [34] to obtain effective densities of obstacles that can cause blockage (also known as densities of obstacles in the blocking region). For instance, in [7], [8], [13], this method is used to derive and analyze outage and coverage probabilities
for mmWave cellular networks. In [30], stochastic geometry is used to analyze the capacity of mmWave networks. In [35], the authors derive the rate and coverage probability for outdoor cellular networks using a ball-based blockage model. However, despite the recent progress in studying the effect of blockage on rate and coverage [7], [8], [12], [29], [35], there are no works in the literature on the effect of blockage on the error performance of STCs. Table I summarizes the novel contributions of this paper with respect to previous studies.

Our novel contributions are summarized as follows.

- We derive upper and lower bounds on the diversity and coding gains of any STC over the Nakagami-$m$ fading channel. We then identify necessary and sufficient conditions for STCs to achieve the upper bounds, which leads to a fundamental trade-off between maximizing the diversity and coding gains. We also propose design guidelines for STCs operating over the Nakagami-$m$ fading channel.

- We show that certain classes of STCs satisfy the derived conditions and, hence, achieve the optimal diversity and coding gains, whereas STCs that are otherwise optimal for the Rayleigh and Rician fading channels do not achieve the optimal diversity and coding gains over the Nakagami-$m$ channel. The theoretical analysis is further supported with numerical simulations of appropriate STCs.

- We then analyze the impact of employing a large number of antennas on the error performance of STCs over Nakagami-$m$ fading channels. Additional design criteria are

### TABLE I: A summary comparison of the contributions of this paper with respect to previous studies is shown.

<table>
<thead>
<tr>
<th>Source</th>
<th>Performance Metric(s)</th>
<th>Multipath Fading</th>
<th>Effect of Employing a Large Number of Antennas</th>
<th>Effect of Blockage</th>
<th>Obstacles’ Heights and Diameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tarokh et al. [21]</td>
<td>PEP</td>
<td>Rayleigh, Rician</td>
<td>Not Analyzed</td>
<td>Not Analyzed</td>
<td>N/A</td>
</tr>
<tr>
<td>Jafarkhani [27]</td>
<td>PEP</td>
<td>Rayleigh, Rician</td>
<td>Analyzed</td>
<td>Not Analyzed</td>
<td>N/A</td>
</tr>
<tr>
<td>Mareef and Aissa [19]</td>
<td>PEP (OSTBCs only)</td>
<td>Nakagami-$m$</td>
<td>Not Analyzed</td>
<td>Not Analyzed</td>
<td>N/A</td>
</tr>
<tr>
<td>Bai and Heath [7]</td>
<td>Rate and Coverage</td>
<td>Nakagami-$m$</td>
<td>Not Analyzed</td>
<td>Analyzed</td>
<td>Fixed</td>
</tr>
<tr>
<td>Chelli et al. [10]</td>
<td>BEP (Multihop Relay)</td>
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</tr>
<tr>
<td>Bellbase et al. [14]</td>
<td>Outage (Multihop Relay)</td>
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<td>Not Analyzed</td>
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<td>Fixed</td>
</tr>
<tr>
<td>Kusaladharma et al. [15]</td>
<td>Outage (D2D)</td>
<td>Nakagami-$m$</td>
<td>Not Analyzed</td>
<td>Analyzed</td>
<td>Variable</td>
</tr>
<tr>
<td>Yang et al. [11]</td>
<td>Delay Analysis</td>
<td>Nakagami-$m$</td>
<td>Not Analyzed</td>
<td>Not Analyzed</td>
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</tr>
<tr>
<td>Fatnassi and Rezki [13]</td>
<td>Rate and Coverage</td>
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<td>Not Analyzed</td>
<td>Analyzed</td>
<td>Variable</td>
</tr>
<tr>
<td>This Paper</td>
<td>PEP</td>
<td>Nakagami-$m$</td>
<td>Analyzed</td>
<td>Analyzed</td>
<td>Variable</td>
</tr>
</tbody>
</table>
provided for three general cases: a large number of transmit antennas, a large number of receive antennas, and a large number of transmit and receive antennas.

- To find the fundamental impact of blockage on the error rates of mmWave channels, we use stochastic geometry to model a typical indoor environment with random obstacles that obstruct the signals between two transceivers. Interestingly, our analysis and simulations indicate that blockage mainly reduces the coding gain but does not impact the diversity gain. The reduction in the coding due to blockage is given as a function of the coding gain without blockage and the environmental parameters dictating blockage.

The remainder of this paper is outlined as follows. In Section II, we present the system model. We derive the diversity and coding gains in Section III. We then validate our theoretical results with numerical simulations in Section IV. Finally, we conclude the study and point out new research directions in Section V.

**Notation:** We represent an arbitrary $N \times M$ matrix $A$ with complex entries $a_{i,j}$ by $A = [a_{i,j}] \in \mathbb{C}^{N \times M}$. The square of the Frobenius norm of an arbitrary matrix $A = [a_{i,j}] \in \mathbb{C}^{N \times M}$ is defined as $\|A\|_F^2 \triangleq \text{tr}(A^H A)$, where $\text{tr}(A)$ represents the trace of $A$, and $A^H$ denotes its Hermitian. $\Pr\{\zeta\}$ denotes the probability measure over some sample space $\Omega$ of a subset $\zeta \subset \Omega$.

We represent a random variable by an uppercase letter, e.g., $X$, its realization by a lowercase letter, e.g., $x$, and its probability density function (PDF) by $f_X(x)$. $\mathbb{E}_X\{\cdot\}$ denotes the expectation of the expression inside the braces over random variable $X$; we omit the subscript when the expectation is taken over all random variables inside the braces. We represent a Nakagami-distributed random variable $X$ with parameters $m \geq 1/2$ and $\Omega > 0$ by $X \sim \text{Nakagami}(m, \Omega)$, a random variable $Y$ that is uniformly distributed over an interval $(a, b) \subset \mathbb{R}$ by $Y \sim U(a, b)$, and a random variable $Z$ that is Poisson-distributed with density $\lambda > 0$ by $Z \sim \text{Poisson}(\lambda)$. We denote the amplitude and phase of a complex number $h \in \mathbb{C}$ by $|h|$ and $\angle h$, respectively. We write $f(\rho) \doteq g(\rho)$ to denote that $\lim_{\rho \to \infty} \log(f(\rho))/\log(g(\rho)) = 1$, and $\preceq$ is defined similarly. We write $f(\rho) \ll g(\rho)$ to denote that $\lim_{\rho \to \infty} f(\rho)/g(\rho) = 0$.

**II. System Model**

Any wideband channel can be split into a set of parallel narrowband channels using orthogonalization techniques [36]; hence, we assume a narrowband channel. Under the narrowband assumption, the realizations of the channel transfer function are constant over all frequencies such
that frequency components of transmitted signals experience the same multiplicative, complex gain. As a result, the convolution of the time-domain representation of the transmitted signal and the channel impulse response simplifies to a multiplication of the transmitted signal by a constant. We also assume the realization of the channel matrix is constant over the length of the code block (i.e., quasi-static fading), and the receiver has perfect channel state information (CSI-R). These assumptions are further justified and discussed in [7], [8], [21], [22], [27], [37].

For a MIMO system with $N_t$ and $N_r$ transmit and receive antennas, respectively, and considering a block code of length $T$, the received symbols matrix $\mathbf{R} = [r_{i,j}] \in \mathbb{C}^{N_r \times T}$ is given by:

$$\mathbf{R} = \sqrt{\rho} \Gamma \mathbf{H} \mathbf{S} + \mathbf{Z},$$  \hspace{1cm} (1)

where $\mathbf{H} = [h_{i,j}] \in \mathbb{C}^{N_r \times N_t}$ is the normalized channel matrix, where $|h_{i,j}| \sim $ Nakagami($m, \Omega$), and $\angle h_{i,j} \sim U(-\pi, \pi)$; $\mathbf{S} = [s_{i,j}] \in \mathbb{C}^{N_t \times T}$ is the normalized transmitted symbols matrix, where each transmitted symbol $s_{i,j}$ belongs to the alphabet $\mathcal{M}$, i.e., $s_{i,j} \in \mathcal{M}$; $\mathbf{Z} = [z_{i,j}] \in \mathbb{C}^{N_r \times T}$ is the AWGN matrix with i.i.d. entries and $z_{i,j} \sim \mathcal{CN}(0, 1)$; $\rho \in [0, \infty)$ is the average signal-to-noise ratio (SNR) at the transmitter and is given by:

$$\rho = \frac{E_b}{N_0} \log_2(M),$$  \hspace{1cm} (2)

where $E_b/N_0$ is the bit energy to noise ratio, and $M = |\mathcal{M}|$, and $\Gamma \in [0, 1]$ is the blockage attenuation parameter that penalizes a LOS communication with probability (w.p.) $P_{LOS}$ and a non-line-of-sight (NLOS) communication w.p. $1 - P_{LOS}$ as:

$$\Gamma = \begin{cases} 
\gamma_1 \triangleq \left( \frac{d}{d_{ref}} \right)^{-\alpha_L} & \text{w.p. } P_{LOS}, \\
\gamma_2 \triangleq \left( \frac{d}{d_{ref}} \right)^{-\alpha_N} & \text{w.p. } 1 - P_{LOS},
\end{cases}$$  \hspace{1cm} (3)

where $d$ is the distance between the transmitter and the receiver, $d_{ref}$ is a reference distance, $\alpha_L$ and $\alpha_N$ are exponent decays experienced by the transmitted signals during LOS and NLOS communications, respectively.

Blockage is a significant phenomenon to consider in highly directive channels, such as the mmWave channel [2], [5], [7], [8]. Hence, we use stochastic geometry to model the statistics of obstacles in indoor environments.

We fix the locations of the transmitter and receiver at heights $h_t$ and $h_r$, respectively, and at a distance $d$ apart. We are interested in the shortest distance $D_0$, between the transmitter
Fig. 1: A blockage event is shown. The occurrence of obstacles is modeled using a PPP. Obstacles are assumed to have random heights and diameters. Blockage happens when an obstacle interrupts the LOS link between the transmitter and the receiver.

and an obstacle that can cause blockage. Due to the assumption that the occurrence of obstacles between the transmitter and receiver follows a Poisson distribution, the distance $D_0$ is uniformly distributed, i.e., $D_0 \sim U(0, d)$ [34]. We assume obstacles are cylindrically-shaped with heights $H_i$ and radii $R_i$ that are uniformly distributed within a certain range, i.e., $H_i \sim U(h_{\text{min}}, h_{\text{max}})$ and $R_i \sim U(r_{\text{min}}, r_{\text{max}})$, where $1 \leq i \leq n$ and $n$ is the number of obstacles between the transceivers. The number of obstacles in a room $n$ is modeled via a PPP with density $\lambda_b$ (number of obstacles per unit volume), i.e., $N \sim \text{Poisson}(\lambda_b)$ [7], [8], [13], [34]. Figure 1 illustrates a realization of the blockage setup described.

Note that the blockage model herein is similar to the blockage model in [7] in the sense that both models consider obstacles using a PPP. The models differ in two aspects. First, in [7], obstacles are buildings, and coverage and rate are analyzed for a cellular system, whereas, in Section IV, obstacles are possibly humans or other objects in an indoor environment, and the effect of blockage on the PEP is investigated. Second, in [7], obstacles are rectangular-shaped, which is appropriate for modeling buildings, whereas, herein obstacles are cylindrical-shaped, which is appropriate for modeling humans and other objects in an indoor environment.
as discussed in detail in [13]. Third, compared with the blockage model in [38], which was developed for mmWave channels in indoor environments, the blockage model in [38] assumes that the heights and the diameters of obstacles are fixed, i.e., all the obstacles have the same height and diameter, whereas the blockage model herein assumes obstacles have random heights and diameters.

### III. Analysis and Results

#### A. Upper Bound on the Conditional PEP

For the system in (1), the PEP is defined as the probability of obtaining codeword $E = [e_{i,j}] \in \mathbb{C}^{N_t \times T}$ at the receiver when codeword $S$ was transmitted. Assuming $R$ is decoded using the maximum likelihood estimator, it can be shown using straightforward algebraic manipulations that the conditional PEP on $H$ and $\Gamma$ is given by [21], [22], [37]:

$$\Pr\{S \rightarrow E|H, \Gamma\} = Q\left(\sqrt{\frac{\rho \Gamma^2}{2}}\|D\|_F\right),$$

where $D \equiv H(S - E)$, $Q(\cdot)$ is the Q-function. Using the Chernoff bound on the Q-function, the conditional PEP in (4) can be upper bounded as [37]:

$$\Pr\{S \rightarrow E|H, \Gamma\} \leq \frac{1}{2} \exp\left(-\frac{\rho \Gamma^2\|D\|_F^2}{4}\right).$$

It is then not difficult to show that $\|D\|_F^2$ can be written as [37, pp. 138–140], [21]:

$$\|D\|_F^2 = \sum_{i=1}^{N_r} \sum_{j=1}^{r} \lambda_j |\beta_{i,j}|^2,$$

where $\{\lambda_j\}_{j=1}^r$ and $r$ are the non-zero eigenvalues and the rank of $(S - E)(S - E)^H$, respectively, and $\beta_{i,j}$ is defined as:

$$\beta_{i,j} \equiv \sum_{j'=1}^{N_t} u_{j',j} h_{i,j'},$$

where $U = [u_{i,j}] \in \mathbb{C}^{N_t \times N_t}$ is unitary (i.e., $UU^H = U^H U = I$), resulting from the singular value decomposition $(S - E)(S - E)^H = U\Lambda U^H$.

Hence, by substituting (6) into (5), an upper bound on the conditional PEP can be written as:

$$\Pr\{S \rightarrow E|H, \Gamma\} \leq \frac{1}{2} \exp\left(-\frac{\rho \Gamma^2}{4} \sum_{i=1}^{N_r} \sum_{j=1}^{r} \lambda_j |\beta_{i,j}|^2\right)$$

(8)
\[
= \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \exp \left( -\frac{\rho \Gamma^2}{4} \lambda_j |\beta_{i,j}|^2 \right). \tag{9}
\]

To get a closed-form expression for an upper bound on PEP under Nakagami-\(m\) fading, we need to average (9) over the channel coefficients.

**B. PDF of the Envelope of a Linear Combination of Complex Nakagami-\(m\) Variables**

To obtain an intuitive understanding from a tight upper bound on the PEP in terms of the diversity gain for MIMO systems under Nakagami-\(m\) fading channel, we approximate the PDF of \(|\beta_{i,j}|\) using a Nakagami-\(m\) distribution with parameters \(\tilde{m}\) and \(\tilde{\Omega}\) as defined in [39], [40]. For \(|h_{i,j}| \sim \text{Nakagami}(m, \Omega)\), \(|\beta_{i,j}|\) is the envelope of the weighted sum of independent, complex Nakagami-\(m\) random variables, where each term in the summation has an amplitude of \(|u_{i,j}h_{i,j}| \sim \text{Nakagami}(m, |u_{i,j}|^2\Omega)\) and a phase of \((\angle h_{i,j} + \angle u_{i,j}) \sim U(-\pi + \angle u_{i,j}, \pi + \angle u_{i,j})\).

Therefore, by applying [39, eq. (98)], [41, eq. (17)], we have that \(|\beta_{i,j}| \sim \text{Nakagami}(\tilde{m}_j, \tilde{\Omega})\) where (cf. the discussion in Appendix A):

\[
\tilde{\Omega} = \Omega, \quad \tilde{m}_j = \frac{m}{(1 - m) \sum_{j=1}^{N_t} |u_{j',j}|^4 + m}, \tag{10}
\]

The expression for \(\tilde{m}_j\) in (10) depends on the codeword choice through the elements of matrix \(U\). To find bounds on \(\tilde{m}_j\) that are independent of the codeword and useful for inferring design criteria, we bound \(\sum_{j'=1}^{N_t} |u_{j',j}|^4\) for all \(u_{j',j} \in \mathbb{C}\) such that \(\sum_{j'=1}^{N_t} |u_{j',j}|^2 = 1\), (using the Cauchy-Schwarz inequality to obtain the non-zero lower bound):

\[
0 < \frac{1}{N_t} \leq \sum_{j'=1}^{N_t} |u_{j',j}|^4 \leq \sum_{j'=1}^{N_t} |u_{j',j}|^2 = 1. \tag{11}
\]

Hence, from the inequality in (11), \(\tilde{m}_j\) is bounded in \(\left( m, \frac{N_t m}{1 + (N_t - 1)m} \right) \) for \(1/2 \leq m < 1\), and is bounded in \(\left[ \frac{N_t m}{1 + (N_t - 1)m}, m \right) \) for \(m \geq 1\). Let \(m_{\text{max}} \triangleq \max \left( \frac{N_t m}{1 + (N_t - 1)m}, m \right)\) and \(m_{\text{min}} \triangleq \min \left( \frac{N_t m}{1 + (N_t - 1)m}, m \right)\). Then, for \(m \geq 1/2\), the expression for \(\tilde{m}_j\) in (10) can be bounded by:

\[
m_{\text{min}} \leq \tilde{m}_j \leq m_{\text{max}}. \tag{12}
\]

We will find the bounds in (12) useful for deriving design criteria in Subsection III-D.
C. Upper Bound on PEP under Nakagami-\textit{m} Fading for MIMO Systems

We find it instructive to first present the treatment of the PEP under Nakagami-\textit{m} fading without blockage and postpone the discussion of the impact of blockage. Thus, we assume for now that $\Gamma = \gamma = 1$, w.p. 1, and delay the averaging over $\Gamma$ in (3) and the analysis of blockage events until Subsection III-H.

Averaging over $|\beta_{i,j}|$ in (9), and for $h_{i,j}$’s independent and identically distributed (i.i.d.), we obtain:

$$\Pr\{S \rightarrow E\} \leq \mathbb{E}\left\{\frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \exp\left(-\frac{\rho}{4} \lambda_j |\beta_{i,j}|^2\right)\right\}$$

(a)$$= \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \mathbb{E}\left\{\exp\left(-\frac{\rho}{4} \lambda_j |\beta_{i,j}|^2\right)\right\}$$

(b)$$= \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \int_{0}^{\infty} \exp\left(-\frac{\rho}{4} \lambda_j x^2\right) \frac{2\tilde{m}_j}{\Gamma(\tilde{m}_j)} x^{2\tilde{m}_j-1} \exp\left(-\frac{\tilde{m}_j x}{\tilde{\Omega}}\right) dx$$

(c)$$= \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \frac{1}{\tilde{\Omega}^{\tilde{m}_j}} \left(\frac{\rho}{4}\lambda_j + \frac{\tilde{m}_j}{\tilde{\Omega}}\right)^{-\tilde{m}_j},$$

where (a) follows from that $\beta_{i,j}$’s are functions of statistically independent random variables and are, hence, also independent (but not necessarily identically distributed); (b) follows from applying the definition of the PDF of $|\beta_{i,j}|$, $f_{|\beta_{i,j}|}(x)$; and (c) follows from the fact that $\int_{0}^{\infty} x^n \exp(-ax^2) dx = \Gamma(\frac{n+1}{2}) / (2a^{\frac{n+1}{2}})$, for $a > 0$ and $n > -1$ [42, eq. (3.326)].

Since, in the high SNR regime, $(\rho/4)\lambda_j \gg \tilde{m}_j/\tilde{\Omega}$, we can upper bound the PEP as:

$$\Pr\{S \rightarrow E\} \leq \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \frac{\tilde{m}_j}{\tilde{\Omega}^{\tilde{m}_j}} \left(\frac{\lambda_j}{\tilde{m}_j} + \frac{4}{\rho}\right)^{-\tilde{m}_j},$$

(17)

$$= \frac{1}{2} \left(\prod_{j=1}^{r} \left(\frac{\lambda_j}{\tilde{m}_j} \sum_{j=1}^{r} \tilde{m}_j\right) \frac{\tilde{\Omega}}{4}\right)^{-N_r \sum_{j=1}^{r} \tilde{m}_j}.$$  

(18)

To deduce coding and diversity gains from (18), we use the definitions of the diversity gain $G_d$ and coding gain $G_c$ in [21, eq. (10)], [43, eq. (4.8)], resulting in the following diversity and coding gains:

$$G_d = N_r \sum_{j=1}^{r} \tilde{m}_j,$$  

(19)
Under Rayleigh fading \((m = 1)\), \(\tilde{m}_j = 1\) for all \(j = 1, \ldots, r\), and we retrieve the expressions for \(G_d\) and \(G_c\) from (19) and (20) that were first derived in [21], [22], i.e., \(G_d = rN_r\), and \(G_c = \frac{\Omega}{4} \prod_{j=1}^{r} \lambda_j^{1/r}\).

Assuming fading is possibly less or more severe than Rayleigh fading (i.e., \(m \neq 1\)), what are the bounds on the diversity and coding gains, and what properties of STCs allow achieving the upper bounds? In the next subsection, we address these questions and extend the design criteria in [21], [22] for Nakagami-\(m\) fading, MIMO channels.

**D. Design Criteria**

The fading severity under Nakagami-\(m\) fading is captured by \(m \geq 1/2\). We want to find properties of STCs that allow for exploiting reduced fading severity to maximize \(G_d\) and \(G_c\). To this end, we use the bounds on \(\tilde{m}_j\) in (12) to bound \(G_d\) and \(G_c\) in (19) and (20), respectively, which results in the bounds in the following proposition.

**Proposition 1:** The achievable gains \(G_d\) and \(G_c\) are bounded by:

\[
rN_r m_{\text{min}} \leq G_d \leq rN_r m_{\text{max}},
\]

and

\[
\frac{\Omega}{4} \prod_{j=1}^{r} \left( \frac{\lambda_j}{m_{\text{max}}} \right)^{\frac{1}{r}} \leq G_c \leq \frac{\Omega}{4} \prod_{j=1}^{r} \left( \frac{\lambda_j}{m_{\text{min}}} \right)^{\frac{1}{r}},
\]

respectively.

**Proof:** The bounds on \(G_d\) follow by a straightforward application of the bounds on \(\tilde{m}_j\) in (12) to the expression of \(G_d\) in (19). The proof of the achievable bounds on \(G_c\) is detailed in Appendix B. It is established by first defining \(x_j \triangleq \tilde{m}_j/\lambda_j\) with the probability mass function (PMF) \(\Pr\{X = \tilde{m}_j/\lambda_j\} = \tilde{m}_j/\sum_{j=1}^{r} \tilde{m}_j\). Then, by taking the logarithm of (20). Maximizing \(G_c\) can be viewed as an entropy maximization problem over the PDF of \(X\), leaning to that \(\Pr\{X = \tilde{m}_j/\lambda_j\} = p\) for all \(j\) since the uniform distribution maximizes the entropy for discrete variables. Hence, with some straightforward algebra, the choice \(\tilde{m}_j = m_{\text{min}} (\tilde{m}_j = m_{\text{max}})\), for all \(j\), achieves the upper (lower) bound in (22). 

\[\blacksquare\]
TABLE II: Properties of $U$ for achieving the bounds on $G_d$ and $G_c$.

<table>
<thead>
<tr>
<th></th>
<th>$m &gt; 1$</th>
<th>$1/2 \leq m &lt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. $G_d$, Min. $G_c$</td>
<td>$U =$ column permutation of the identity matrix</td>
<td>$</td>
</tr>
<tr>
<td>Min. $G_d$, Max. $G_c$</td>
<td>$</td>
<td>u'_{j,j}</td>
</tr>
</tbody>
</table>

To achieve the bounds in (21) and (22) on $G_d$ and $G_c$, codewords must produce a matrix $U$ that satisfies the properties in the following proposition.

**Proposition 2:** For a given $r$, achieving the upper bound on $G_d$ results in achieving the lower bound on $G_c$, and vice versa. In particular, each bound is achieved if and only if $U$ satisfies the respective properties in Table II.

**Proof:** For convenience, the proof is presented in Appendix C. It is established by maximizing $\tilde{m}_j$ in (10) over $|u'_{j,j}|$. Thus, first, the term $\sum_{j'=1}^{N_t} |u'_{j,j}|^4$ is lower and upper bounded using the Chauchy-Schwarz inequality. Then, from the unitary constraint on $U$, it is not difficult to show that the properties in Table II follow.

Note that there exist codes that satisfy the conditions on $U$ in Table I. Exemplary codes that satisfy these conditions for different values of $m$ are given in Section IV.

From Proposition 2, we can illustrate STCs that achieve maximum gains and infer design criteria as follows.

- **OSTBCs** produce a unitary matrix $U$ that is the identity matrix [37, pp. 145–147]. Therefore, for $m \geq 1$, OSTBCs satisfy Proposition 2 and, hence, achieve the upper bounds on $G_d$ of $rN_t m_{max}$. Note that, for $1/2 \leq m < 1$, OSTBCs do not achieve the maximum diversity gain as predicted by Proposition 2.

- To maximize the diversity gain of any STC that achieves the upper bound on the diversity gain of $rN_t m_{max}$, designers must maximize $r$, the $r$, which is also known as the rank criterion [21].

- For a given $N_t$, since $r \leq \min(N_t, T)$, the length of the codeword $T$ should be sufficiently large, i.e., $T \geq N_t$. To overcome excessive decoding delays, $T$ must be chosen to be equal
TABLE III: Main differences between our proposed and conventional design criteria.

<table>
<thead>
<tr>
<th></th>
<th>Closed-Form Upper Bound</th>
<th>Determinant Criterion</th>
<th>Codes that Achieve Full Diversity</th>
<th>Trade-off between $G_d$ and $G_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tarokh et al. [21]</td>
<td>Rayleigh, Rician</td>
<td>Always applies</td>
<td>Full-rank Codes</td>
<td>No</td>
</tr>
<tr>
<td>This Paper</td>
<td>Nakagami-$m$</td>
<td>Special case</td>
<td>Full-rank codes that satisfy Proposition 2</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Remark 1: Though, in [21], the analysis shows that the determinant criterion always holds for the Rayleigh and Rician fading channels, this is not necessarily true for the Nakagami-$m$ fading channel. The reason is that maximizing $\prod_{j=1}^{r} \lambda_j$ (which is the determinant criterion) does not necessarily maximize $G_c$ in (20). Table III summarizes the main differences between our derived design criteria and the ones derived in [21].

Remark 2: The parameter $m$ measures the severity of multipath fading. For $1/2 \leq m \leq 1$, the Nakagami-$m$ distribution corresponds to the Nakagami-$q$ (Hoyt) distribution [39, p. 18], which models fading scenarios where fading is more severe than Rayleigh fading. Whether such deep fades occur over the mmWave channel depends on the environment (indoor or outdoor) and the propagation frequency. In [6], the Nakagami-$q$ (Hoyt) distribution is considered for modeling deep fade scenarios over the mmWave channel. The more generalized model in [6] is shown to provide an excellent fit to measurements taken in an outdoors urban environment at 28 GHz.
E. Employing a Relatively Large Number of Receive Antennas

First, we analyze the effect of having a large number of receive antennas \((N_r \to \infty)\), while the number of transmit antennas \(N_t\) is fixed.

**Proposition 3:** For finite \(N_t\), as \(N_r \to \infty\), we have that

\[
\Pr\{S \to E\} \leq \frac{1}{2} \left( \prod_{j=1}^{r} \exp \left( \frac{\Omega \log_2(M) E_b}{4 N_t N_0 \lambda_j} \right) \right)^{-N_r}.
\]  

**Proof:** For convenience, the proof is presented in Appendix D. First, the law of large numbers is invoked to show that \(\lim_{N_r \to \infty} \|D\|_F^2/\Omega \sum_{j=1}^{r} \lambda_j = 1\). Then, properties of the limit are used to establish (23).

From (23), we observe that the PEP goes exponentially to zero in \(E_b/N_0\), when \(N_r\) is very large. Note that this is not the case for finite \(N_r\) where the PEP decreases as \((E_b/N_0)^{-N_r}\) which is much slower than \(\exp(E_b/N_0)^{-N_r}\). The design criterion in this case is to maximize the product \(\prod_{j=1}^{r} \exp \left( \frac{\Omega}{4} \lambda_j \right)\). This is in turn equivalent to maximizing the sum of eigenvalues \(\lambda_j\), which is also known as the trace criterion \[27\]. Note that the trace criterion is less stringent than the determinant criterion in the sense that, if one eigenvalue \(\lambda_i\) is small but nonzero, then the resulting product of eigenvalues \(\prod_{j=1}^{r} \lambda_j\) is also small, and hence the determinant is small. But, even if there exists one small but nonzero eigenvalue \(\lambda_i\), the sum of eigenvalues \(\sum_{j=1}^{r} \lambda_j\) could still be large.

F. Employing a Relatively Large Number of Transmit Antennas

Next, we consider the effect of having \(N_t \to \infty\), while \(N_r\) is fixed. To this end, we recall that \(|\beta_{i,j}| \sim \text{Nakagami}(\tilde{m}_j, \tilde{\Omega})\) (cf. Subsection III-B), and observe the effect of having \(N_t \to \infty\) on \(\tilde{m}_j\) in (10). We would like to maximize the diversity and coding gains. Note that, unlike the previous case, since \(N_r\) is is fixed, we cannot use (54). It is helpful to distinguish between two main cases that depend on \(m\).

**Case 1:** \(m > 1\)

To achieve the maximum \(G_d\), according to Proposition 2, we must have that \(\sum_{j'=1}^{N_t} |u_{j',j}|^4 = 1\). Hence, from (10), \(\tilde{m}_j = m\) for all \(j\), regardless of how large \(N_t\) is. Since \(r \leq \min(T, N_t)\), as \(N_t \to \infty\) and for a finite \(T, r \leq T\). Therefore, the maximum \(G_d\) is \(TN_t m\). Given that the maximum \(G_d\) is achieved, one can only achieve the minimum \(G_c\), in accordance with Proposition
2. To maximize the achievable $G_c$, the conventional determinant criterion holds (cf. our general discussion in Subsection III-D).

**Case 2:** $1/2 \leq m < 1$

To achieve the maximum $G_d$, from Proposition 2, the design requirement is that $\sum_{j'=1}^{N_t} |u_{j',j}|^4 = 1/N_t$, for all $1 \leq j \leq N_t$. Then, as $N_t \to \infty$, $\sum_{j'=1}^{N_t} |u_{j',j}|^4 \to 0$; hence, from (10), $\tilde{m}_j \to 1$, and $|\beta_{i,j}|$ converges in distribution to a Rayleigh envelope. Therefore, for finite $T$, $G_d$ is limited to $TN_r$. Furthermore, to maximize $G_c$, the conventional determinant criterion applies.

**G. Employing a Relatively Large Number of Transmit and Receive Antennas**

We consider the case in which both $N_t$ and $N_r$ are large but their ratio is fixed.

**Proposition 4:** For $N_r/N_t = a$, where $0 < a < \infty$, as $N_r \to \infty$:

$$\Pr\{S \to E\} \leq 2 \left( \prod_{j=1}^{r} \exp \left( \frac{\Omega \log_2(M) E_b}{4 N_0 \lambda_j} \right) \right)^{-a}. \quad (24)$$

**Proof:** For convenience, the proof is presented in Appendix E. The proof follows similar steps as those used in the proof of Proposition 3.

From (24), the PEP goes to zero in $E_b/N_0$ as $\exp(E_b/N_0)^{-a}$ which, for large enough $a$, approaches zero faster than $N_r^{-N_r}$, which is the PEP decay rate for finite $N_t$ and $N_r$. In this case, the trace criterion holds, i.e., we would like the codeword to maximize the sum of eigenvalues $\{\lambda_j\}_{j=1}^{r}$. A strategy for satisfying the trace (determinant) criterion is described as follows. When designing a STC, we look at the difference between any two different codewords, say $S_i$ and $S_j$ ($i \neq j$), and we simply maximize the minimum trace (determinant) of $(S_i - S_j)(S_i - S_j)^H$ over all $i$ and $j$, where $i, j = 1, \ldots, K$, where $K$ is the number of codewords in the space-time coding scheme.

An interesting remark about the benefits of having a large number of transmit and receive antennas is in order. The PEP decays to zero exponentially in the cases where the number of receive antenna is large and where the number of transmit and receive antennas is large, but not when only the number of transmit antennas is large. This means that when it is desirable to have the PEP go to zero exponentially in the SNR, designers should increase the number of transmit antennas or increase the number of transmit and receive antennas at a fixed ratio.
Note that in practical wireless channels with large number of antennas issues such as pilot contamination and hardware misalignment must be considered. In our treatment, we focused solely on the PEP due to space limitation and to highlight the effect of employing a large number of antennas on the diversity and coding gains.

**H. Averaging the PEP over the Blockage Attenuation Factor $\Gamma$**

The PEP conditioned on $\Gamma$ is given by:

$$
\Pr\{S \rightarrow E | \Gamma\} = 1 - \frac{1}{2} N_r \prod_{i=1}^{r} \prod_{j=1}^{\tilde{m}_j} \left(\frac{\tilde{m}_j / \tilde{\Omega}}{(\rho / 4) \lambda_j \Gamma^2 + \tilde{m}_j / \tilde{\Omega}}\right)^{\tilde{m}_j}.
$$

(25)

Averaging (25) over $\Gamma$, gives:

$$
\Pr\{S \rightarrow E\} = \mathbb{E}_{\Gamma}\{\Pr\{S \rightarrow E | \Gamma\}\}
= \frac{1}{2} N_r \prod_{i=1}^{r} \prod_{j=1}^{\tilde{m}_j} \left(\frac{\tilde{m}_j / \tilde{\Omega}}{(\rho / 4) \lambda_j \gamma^2 + \tilde{m}_j / \tilde{\Omega}}\right)^{\tilde{m}_j} P_{\text{LOS}} + \frac{1}{2} N_r \prod_{i=1}^{r} \prod_{j=1}^{\tilde{m}_j} \left(\frac{(\rho / 4) \lambda_j \gamma^2 + \tilde{m}_j / \tilde{\Omega}}{\tilde{m}_j P_{\text{LOS}} + \frac{1}{2} N_r \prod_{i=1}^{r} \prod_{j=1}^{\tilde{m}_j} \left(\frac{\tilde{m}_j / \tilde{\Omega}}{(\rho / 4) \lambda_j \gamma^2 + \tilde{m}_j / \tilde{\Omega}}\right)^{\tilde{m}_j}}\right)^{\tilde{m}_j}(1 - P_{\text{LOS}}).
$$

(26)

To see the impact of blockage on the diversity and coding gains, we express $P_{\text{LOS}}$ as a function of known parameters such as the density of obstacles and their dimensions and the transceivers’ locations.

**I. Expressing $P_{\text{LOS}}$ as a Function of Environmental Properties**

We want to find $P_{\text{LOS}}$ as a function of the environmental properties. To this end, we first find the effective density of obstacles. The density of obstacles in the room is $\lambda_b$, but not all obstacles present in a room can cause blockage. Using the Thinning Theorem from stochastic geometry [34], the effective density of obstacles that can cause blockage, denoted by $\lambda'_b$, can be expressed as:

$$
\lambda'_b = \lambda_b \Pr\{\xi_1\} \Pr\{\xi_2\},
$$

(28)

where $\Pr\{\xi_i\} (i = 1, 2)$ are blockage events that need to occur simultaneously for the LOS link to be blocked. In particular, $\xi_1$ ($\xi_2$) is the event that an obstacle is tall (wide) enough to obstruct the LOS.

To find $\Pr\{\xi_1\}$, consider Fig. 1(a). For $n$ obstacles in the blocking region with heights $H_1, H_2, \ldots, H_n$, let $H_0 \triangleq \max (H_1, H_2, \ldots, H_n)$ be the height of the tallest obstacle between
the transmitter and receiver. From Fig. 1(a), an obstacle with height $H_0$ and distance $D_0$ from the transmitter can block the LOS $h(x)$ if and only if $H_0 > h(D_0)$, where $h(x)$ is the LOS. Hence, as we show in more detail in Appendix F, $\Pr\{\xi_1\}$ is given by:

$$\Pr\{\xi_1\} = \mathbb{E}_{D_0} \left\{ \Pr \left\{ H_0 > h(d_0) \mid D_0 = d_0 \right\} \right\}$$

$$= 1 - \frac{1}{(h_{\text{max}} - h_{\text{min}})^n} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (h_t - h_{\text{min}})^{n-k} \left( \frac{h_t - h_t}{k+1} \right)^k. \quad (29)$$

Finding $\Pr\{\xi_2\}$, follows a similar procedure as illustrated by Fig. 1(b). Only an object with a radius greater than $|y|$ can cause a blockage event, where $y$ is the distance of the line perpendicularly joining the center of the blocking object to the LOS between the transceivers. For $n$ obstacles in the blocking region with radii $R_1, R_2, \ldots, R_n$, let $R_0 \triangleq \max(R_1, R_2, \ldots, R_n)$ be the radius of the widest obstacle between the transmitter and receiver. Then, as shown in Appendix F, $\Pr\{\xi_1\}$ is given by:

$$\Pr\{\xi_2\} = \mathbb{E}_Y \left\{ \Pr \left\{ R_0 > |y| \mid Y = y \right\} \right\}$$

$$= 1 - \frac{1}{(r_{\text{max}} - r_{\text{min}})^n} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) (-r_{\text{min}})^{n-k} \frac{r_0^k}{k+1}. \quad (30)$$

The events $\xi_1$ and $\xi_2$ restrict the volume over which obstacles can cause blockage to $2dh_tr_{\text{max}}$. Thus, the number of obstacles that can cause blockage is $n = \lceil 2\lambda_b dh_tr_{\text{max}} \rceil$.

Now, to evaluate the average PEP in (27), we find the probability of having a LOS link between the transceivers $P_{\text{LOS}}$. Let $K'$ denote the effective number of obstacles that can cause blockage, i.e., $K' \sim \text{Poisson}(\lambda'_b)$. Then, the probability of having a LOS communication is given by:

$$P_{\text{LOS}} = \Pr \{ K' = 0 \} = \exp (- \lambda'_b). \quad (33)$$

We observe from (33) that $P_{\text{LOS}}$ is not a function of the SNR. And, since the diversity gain is defined in terms of the behavior of the PEP as the SNR goes to infinity, the diversity gain is not affected by the presence of obstacles. This makes sense since the diversity gain in MIMO systems is caused by the propagation of multipath components, which involve reflected paths that are not LOS, whereas blockage only involves obstacles obstructing LOS propagation. Notice that the fact that $P_{\text{LOS}}$ is independent of the SNR holds for any fading channel; hence, the diversity gain is not affected by blockage for any fading statistics.
By averaging the PEP, and then averaging the coding gain over \( \Gamma \), we obtain

\[
G^b_c = \left\{ P_{\text{LOS}} \left( \frac{d}{d_{\text{ref}}} \right)^{-\alpha_L} + (1 - P_{\text{LOS}}) \left( \frac{d}{d_{\text{ref}}} \right)^{-\alpha_N} \right\} G_c,
\]

where \( G^b_c \) is the effective coding gain when the effect of blockage is considered, and \( G_c \) is given by (20). We observe from (34) that, for \( \alpha_L = \alpha_N = 0 \), the SNR is not reduced due to either LOS or NLOS communication. Hence, the coding gain with blockage is maximized, i.e., \( G^b_c = G_c \). The expression in (34) holds regardless of the fading statistics. Note that the results concluded regarding the effect of blockage depend on the blockage modeling assumptions, where the occurrence of obstacles follow a PPP. It would be an interesting topic for future research to investigate the effect of different blockage modeling assumptions, e.g., Matern hard-core point process (MHCP) of type I [44]–[46], on the PEP. We consider the treatment here to be a starting point for such interesting analysis.

IV. NUMERICAL RESULTS

In this section, we simulate the error performance of STCs over Nakagami-\( m \) fading channels to illustrate our proposed design criteria and the effect of blockage on the diversity and coding gains. The notation \( N_t \times T \) refers to coding over \( N_t \) transmit antennas and \( T \) channel uses. The average PEP \( P_r\{S \to E\} \) is itself an upper bound on the symbol error probability \( P_s \), see [37,
pp. 136-137]. But, as $\rho \to \infty$, the two error probabilities converge to the same value times a constant $c > 0$ [37, pp. 142-143], i.e., $P_s \approx c \Pr\{S \to E\}$. Furthermore, for a codeword of length $T$ and a modulation order of $M$, $P_b$ is related to $P_s$ as $P_b \approx \frac{P_s}{T \log_2 M}$ [37]. Hence, in the high SNR regime, $P_b \approx \left(\frac{c}{T \log_2 M}\right) \Pr\{S \to E\}$, and we can study the PEP by investigating the symbol and bit error rates. We indicate the asymptotic curves using dashed or dotted lines and
Fig. 5: Effect of large $N_t$ on the SER versus SNR performance.

Fig. 6: Effect of large $N_t$ on the SER versus SNR performance.

(a) $N_r$ and $T$ are fixed, and $N_t$ increases ($m = 1$).  
(b) $N_r$ is fixed, and $m$ increases.

the Monte Carlo simulated curves using solid lines. For simulations that require bit sequences longer than $10^9$ symbols, numerical averaging of (4) was performed over all possible codewords and a large number of realizations of the channel matrix.

Figure 2(a) shows the symbol error rate (SER) for the OSTBC $(2 \times 2)$, which is the Alamouti code [17], and the GC $(2 \times 2)$ [26] over the Nakagami-$m$ fading channel. The modulations used
Fig. 7: Effect of large $N_t$ and $N_r$ on the SER versus SNR performance.

are BPSK and QPSK for the GC $(2 \times 2)$ and the OSTBC $(2 \times 2)$, respectively. This choice of modulations is to normalize the transmission rates in bits per channel use for each scheme since the GC $(2 \times 2)$ is full-rate and the OSTBC $(2 \times 2)$ is rate-1, i.e., now both schemes transmit at the same rate of 2 bits per channel use. When $m = 1$, i.e., Rayleigh fading, both the OSTBC $(2 \times 2)$ and the GC $(2 \times 2)$ achieve the full diversity of $N_tN_r$, which is 4 in this case. However, when the fading severity is reduced, i.e., $m = 2$, the OSTBC $(2 \times 2)$ achieves the full diversity of $mN_tN_r$, which is 8, while the GC $(2 \times 2)$ only achieves a diversity gain of about 5.5. Note that, when $m = 2$, the worst-case $\tilde{m}_j$ for the OSTBC $(2 \times 2)$ is 2, resulting in an achievable diversity gain of 8, and the worst-case $\tilde{m}_j$ the GC $(2 \times 2)$ is 1.333, resulting in an achievable diversity gain of about 5.332. Hence, the reason why orthogonal codes are able to achieve the maximum diversity order of $N_tN_rm_{\text{max}}$, for $m \geq 1$, is that they produce a unitary matrix $U$ that is the identity, as derived in Section III-D. On the other hand, though the GC $(2 \times 2)$ achieves full-diversity over Rayleigh fading channels [26], it does not necessarily result in a $U$ whose columns are permutations of those of the identity; hence, it does not achieve full diversity over Nakagami-$m$ fading channels.

In Fig. 2(b), we simulate the bit error rate (BER) of the OSTBC $(2 \times 2)$ when blockage is incorporated into the channel model. The simulated scenario uses values for $\alpha_L$ and $\alpha_N$ in [28] for 60 GHz and $\gamma_1 = 0.9$ and $\gamma_2 = 0.6$. Varying the density $\lambda_b$ changes $P_{\text{LOS}}$ as given by
We observe from Fig. 2(b) that increasing the probability of a LOS communication does not affect the slope of the BER versus SNR curve but shifts the curve to the left. We represent this shift in the BER curve by $\Delta G_c$. For a BER of about $10^{-3}$, reducing $P_{\text{LOS}}$ from 0.8 to 0.5 and 0.1 shifts the BER curve to left by 1.0 dB and 2.0 dB, respectively. By plugging the mentioned parameters into (34), it can be readily verified that the predicted reduction in the coding gain is by 1.09 dB and 2.1 dB, when decreasing $P_{\text{LOS}}$ from 0.8 to 0.5 and 0.1, respectively, which is consistent with the observed simulation. That is, the coding gain which is represented by translations of the BER curve along the SNR axis is influenced by blockage, whereas the diversity gain which is represented by the slope is not, conforming with our analysis in Section III.

We repeat our simulation for a higher number of transmit and receive antennas as shown in Figures 3. Specifically, we simulated a simple repetition code (RC) $(4 \times 4)$ with four transmit and receive antennas and the OSTBC $(4 \times 4)$ proposed in [22, eq. (4)], i.e., the codeword $\mathcal{D}_{4\times 4}$ in [37, eq. (7.51)]. Figure 3(a) shows that even for a larger number of transmit antennas, a simple RC is unable to improve the diversity gain as we reduce the severity of fading, i.e., as $m$ is doubled. On the other hand, as shown in Fig. 3(a), the OSTBC $(4 \times 4)$ is able to double the diversity gain as we double the value of $m$. More precisely, the diversity gain achieved by the RC $(4 \times 4)$ is 8 regardless of the value of $m$, whereas the diversity gain by the OSTBC $(4 \times 4)$ is 16 and 32 for $m = 1$ and $m = 2$, respectively. In Fig. 3(a), A BER of $10^{-3}$ is achieved at 3.5 dB for a $P_{\text{LOS}}$ of 0.9. As $P_{\text{LOS}}$ decreases to 0.5 and 0.1, the BER of $10^{-3}$ is achieved at about 5 dB and 6 dB, respectively. But, changing $P_{\text{LOS}}$ does not affect the slopes of BER curves. Thus, the coding gain is affected by blockage, but the diversity gain is not. Hence, our design criteria are illustrated to hold for a larger number of transmit antennas as well. Furthermore, increasing $P_{\text{LOS}}$ results only in a shift in the BER curve to the left, increasing the coding gain, and does not affect the diversity gain (slope of the BER curve), as shown in Fig. 3(b).

To further demonstrate our analytical results and design criteria, we simulate error performance of STCs that achieve the bounds on the diversity gain for the case when $0.5 \leq m < 1$ in Fig. 4. As can be seen in Fig. 4(a), the OSTBC $(2 \times 2)$ achieves diversity gains of two and three for $m = 0.5$ and $m = 0.75$, respectively. According to our general discussion in Subsection III-D, for $1/2 \leq m < 1$, OSTBCs can only achieve a diversity gain of $r N_r m_{\text{min}}$. In this case, since $m_{\text{min}} = m$, the achieved diversity gains are $r N_r (1/2) = 2$ and $r N_r (3/4) = 3$ for $m = 0.5$.
and $m = 0.75$, respectively. However, in Fig. 4(b), we consider a new proposed STC that is designed to achieve full diversity for $0.5 \leq m < 1$ based on Proposition 2. More specifically, the new code transmits the codewords $S_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 1 & 1 - \frac{1}{\sqrt{2}} \\ 1 & 1 - \frac{1}{3\sqrt{2}} \end{pmatrix}$ to transmit bits 0 and 1, respectively. This choice of codewords is a possible solution to the system of equations defined by $(S - E)(S - E)^H = U\Lambda U^H$, where $\lambda_1 + \lambda_2 = 1$ (which normalizes the power of codewords) and $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which satisfies the condition in Proposition 2 for achieving full diversity when $1/2 \leq m < 1$, where $\lambda_1 = 2/3$ for the simulated code. It can be easily verified using (10) that this code produces $\tilde{m}_j = m_{\text{max}}$ for $1/2 \leq m < 1$. As shown in Fig. 4(b), the new code achieves diversity gains of 2.66 and 3.43 when $m = 0.5$ and $m = 0.75$, respectively. This corresponds to achieving the full diversity of $rN_r m_{\text{max}}$ when $1/2 \leq m < 1$. Note that the OSTBC $(2 \times 2)$ achieves a higher coding gain than the new code which can be explained by the trade-off between the diversity and coding gains in Proposition 2.

In Fig. 5, we illustrate the effect of having a large number of receive antennas on the error performance. As shown in Fig. 5, when $N_r = 4$, the SER goes to zeros as $(E_b/N_0)^{-rN_r}$. But, for large $N_r$, e.g., $N_r = 64$ and $N_r = 128$, the SER decays as $\exp(-N_r E_b/N_0)$. This is consistent with our analysis for the case in which $N_r$ is very large, in Section III-D. Hence, the benefits of having a large number of receive antennas extend to also having an exponential decrease in the error rate as the SNR increases.

Figure 6 shows the error performance of STCs when the number of transmit antennas is large. In Fig. 6(a), we observe that when $T$ is fixed, the maximum diversity gain of four $(TN_t)$ is achieved both when $N_t = T$ and when $N_T > T$. The reason, as discussed in Section III-D, is that $r \leq T$ for $N_t > T$. Hence, the maximum achievable diversity gain of $rN_r m_{\text{max}}$ is dictated by $T$ for fixed $N_r$ and $m_{\text{max}}$. But, as also shown in Fig. 6(a), increasing $N_t$ can provide lower error rates by increasing the coding gain. Further, Fig. 6(b) emphasizes that when $N_t$ is much larger than $N_r$ and for a fixed $T$, increasing $m$ does not change the diversity gain, as demonstrated by the two approximately parallel curves when $m = 1$ and $m = 4$ at high SNR values. Hence, for $N_t \gg N_r$, the error performance converges to that of a Rayleigh fading channel as the SNR increases, which agrees with our analysis in Section III-D.

In Fig. 7, we illustrate the error performance of STCs when number of antennas at the trans-
mitter and receiver are both large, but their ratio is fixed. We observe that when $N_r = N_t = 16$ and when $N_r = N_t = 32$, the SER goes to zero as $\exp(-E_b/N_0)$, which conforms with our analysis in Section III-D.

V. Conclusion

To provide more generalized design criteria for STCs that apply over mmWave channels, we derived an upper bound on the PEP assuming the channel coefficients modeling small-scale fading are complex random variables with Nakagami-$m$-distributed amplitudes and uniformly distributed phases. The upper-bound describes well the diversity and coding gains over MIMO Nakagami-$m$ fading channels. We deduced design criteria for exploiting the maximum diversity and coding gains and showed new interesting properties of STCs that achieve the maximum gains. Traditional determinant and rank criteria may lead to suboptimal performance when applied to the Nakagami-$m$ fading channel. The more generalized design criteria proposed outlines conditions for achieving optimal error performance over the Nakagami-$m$ fading channel. Furthermore, we showed that there is a trade-off between the diversity and coding gains.

We investigated the effect of having a large number of transmit and receive antennas on the PEP. We showed that employing a large number of antennas has extended benefits in terms of reliability. For instance, we showed that, when the number of receive antennas is large and when the number of transmit and receive antennas is large at a fixed ratio, the PEP goes to zero exponentially in the SNR, whereas, when the number of transmit antennas only is large, the PEP goes polynomially to zero in the SNR. We also studied the effect of blockage due to humans and other objects on the PEP. Our analysis and simulations show that blockage results only in a shift of the BER versus SNR curve (i.e., affects coding gain), and does not affect the slope of the BER curve (i.e., does not affect the diversity gain). We expressed the reduction in the coding gain as a function of the probability of a LOS communication, path loss exponents, the distance between transceivers, and the density of obstacles.

The following are interesting future research directions.

- The design of STCs that satisfy the proposed design criteria and are optimized for Nakagami-$m$ fading channels is an appealing research problem.
- Analysis of the average error performance of STCs when applied over mmWave cellular networks.
- Analysis of the diversity and coding gains when the receiver implements sub-optimal, but more computationally efficient, decoding methods.
- Analyzing them effect of blockage on the PEP for different blockage models.

APPENDIX A

PDF OF THE ENVELOPE OF THE SUM OF COMPLEX NAKAGAMI-\(m\) VARIABLES

To average the expression in (9) over the fading statistics, we first need to find the PDF of \(|\beta_{i,j}| = |\sum_{j'=1}^{N_t} u_{j',j} h_{i,j'}|\). Note that \(|\beta_{i,j}|\) can be viewed as the envelope of the sum of \(N_t\) weighted complex random variables, where the weights have a unity Euclidean norm. Each random variable in this sum has a Nakagami-\(m\)-distributed amplitude and a uniformly distributed phase. An exact, general integral form for the PDF of the magnitude of the sum of independent, complex Nakagami-\(m\) random variables with phases that are statistically independent from amplitudes was first proposed by Nakagami [39]. Later, Du et al. [41] derived this integral form. Nonetheless, Nakagami [39] also proposed that the PDF of the envelope of this sum of Nakagami-\(m\)-distributed random variables is well approximated by another Nakagami-\(m\) distribution with parameters \(\tilde{m}\) and \(\tilde{\Omega}\). This approximation has been empirically shown to match the exact PDF via extensive numerical simulations in [40]. Therefore, the Nakagami-\(m\) distribution has been widely used to approximate the PDF of the envelope of the sum of complex Nakagami-\(m\) distributed in the performance analysis of communication systems [47]. Note that the problem we are considering here is the sum of complex Nakagami-\(m\)-distributed random variables; hence, the known PDF for the sum of real Nakagami-\(m\)-distributed random variables proposed in [48], which is usually applied in performance analysis, cannot be used.

Applying [39, eq. (98)], [41, eq. (17)], we have that \(|\beta_{i,j}| \sim \text{Nakagami}(\tilde{m}_j, \tilde{\Omega})\) where:

\[
\tilde{\Omega} = \sum_{j'=1}^{N_t} |u_{j',j}|^2 \Omega = \Omega, \tag{35}
\]

\[
\tilde{m}_j = \frac{1}{\sum_{j'=1}^{N_t} |u_{j',j}|^4 + m \sum_{j'=1}^{N_t} \sum_{j''=1}^{N_t} |u_{j',j}|^2 |u_{j'',j}|^2}, \tag{36}
\]

\[
= \frac{(a)}{(1 - m) \sum_{j'=1}^{N_t} |u_{j',j}|^4 + m}, \tag{37}
\]

where (a) follows from a straightforward algebraic manipulation and the fact that \(U\) is unitary. Note that, for the case when \(m = 1\) (i.e., Rayleigh fading), \(\tilde{m}_j\) is equal to 1, i.e., the PDF of
\( |\beta_{i,j}| \) is Rayleigh, in agreement with [41].

It is instructive to observe how well the approximation in (37) fits the actual PDF. In Fig. 8, we simulate 10,000 realizations of each \( h_{i,j'} \) for different values of \( m \) and \( N_t \), where \( |u_{j',j}| = 1/\sqrt{N_t} \) for all \( j' \) and \( j \). Generally, the analytical approximation provides a good fit for the Monte Carlo simulated PDF. The approximation improves as \( N_t \) increases.
APPENDIX B

PROOF OF PROPOSITION 1: THE BOUNDS ON THE CODING GAIN

The upper on $G_c$ is obtained as follows. Since $\log(\cdot)$ is a concave function, we can equivalently maximize $G_c$ in (20) by solving:

$$
\max \left\{ \log \left( \prod_{j=1}^{r} \left( \frac{\lambda_j}{\tilde{m}_j} \sum_{j=1}^{m_j} \right) \right) \right\}
$$

(38)

$$
= \max \left\{ \sum_{j=1}^{r} \frac{\tilde{m}_j}{\sum_{j=1}^{m_j}} \log \left( \frac{\lambda_j}{\tilde{m}_j} \right) \right\}
$$

(39)

$$
= \max \left\{ \mathbb{E}_X \left\{ - \log(X) \right\} \right\},
$$

(40)

where $x \triangleq \tilde{m}_j/\lambda_j$, and its PMF is $\Pr\{X = \frac{\tilde{m}_j}{\lambda_j}\} = \frac{\tilde{m}_j}{\sum_{j=1}^{m_j}}$ for all $j$. The expression in (40) can be seen as an entropy maximization problem. It is well-known that the discrete uniform distribution maximizes entropy [36]. Hence, (40) is maximized when $\Pr\{X = \frac{\tilde{m}_j}{\lambda_j}\} = \frac{1}{r}$, and the maximum coding gain is obtained when $\tilde{m}_j = m_{\min}$, i.e.,

$$
\max \left\{ \mathbb{E}_X \left\{ - \log(X) \right\} \right\} = \sum_{j=1}^{r} \frac{1}{r} \log \left( \frac{\lambda_j}{m_{\min}} \right).
$$

(41)

Similarly, to minimize $G_c$ a similar procedure is followed where the dummy random variable is defined as $x \triangleq \frac{\lambda_j}{m_j}$. Then, the minimum $G_c$ would be obtained when $\Pr\{X = \frac{\lambda_j}{\tilde{m}_j}\} = \frac{\tilde{m}_j}{\sum_{j=1}^{m_j} \tilde{m}_j} = \frac{1}{r}$, $\forall j = 1, \ldots, N_t$. Since all $\tilde{m}_j$’s are the same, the minimum is obtained when $\tilde{m}_j = m_{\max}$, i.e., the minimum coding gain is given by:

$$
\min \left\{ \sum_{j=1}^{r} \frac{\tilde{m}_j}{\sum_{j=1}^{m_j} \tilde{m}_j} \log \left( \frac{\lambda_j}{\tilde{m}_j} \right) \right\} = \sum_{j=1}^{r} \frac{1}{r} \log \left( \frac{\lambda_j}{m_{\max}} \right).
$$

(42)

APPENDIX C

PROOF OF PROPOSITION 2: THE TRADE-OFF BETWEEN $G_d$ AND $G_c$

To establish this proof, we first show that:

$$
\frac{1}{N_t} \leq \sum_{j=1}^{N_t} |u_{j',j}|^4 \leq 1.
$$

(43)

The upper bound in (43) follows straightforwardly from the fact that $\sum_{j'=1}^{N_t} |u_{j',j}|^4 \leq \sum_{j'=1}^{N_t} |u_{j',j}|^2 = 1$ (since $|u_{j',j}|^4 \leq |u_{j',j}|^2$ for $0 \leq |u_{j',j}|^2 \leq 1$). To obtain the lower bound in (43), it is instructive
to recall the Cauchy-Schwarz inequality. For two sequences of length $k$, $a = [a_1, \ldots, a_k]$ and $b = [b_1, \ldots, b_k]$ whose elements are complex numbers, we have:

$$\left| \sum_{i=1}^{k} a_i b_i^* \right|^2 \leq \sum_{i=1}^{k} |a_i|^2 \sum_{i=1}^{k} |b_i|^2.$$  \hspace{1cm} (44)

To obtain the lower bound in (43) from (44), take $a_i = |u_{i,j}|^2$ and $b_i = 1$ for all $i$, and let $k = N_t$. Now, we consider what properties of $U$ achieve the upper and lower bounds in (43).

**Achieving the Upper Bound in (43):** When the upper bound is achieved, $\sum_{j'=1}^{N_t} |u_{j',j}|^4 = 1$, $\forall j = 1, \ldots, N_t$, which gives that:

$$\sum_{j'=1}^{N_t} |u_{j',j}|^4 = 1 \iff \sum_{j'=1}^{N_t} |u_{j',j}|^2 (|u_{j',j}|^2 - 1) = 0,$$  \hspace{1cm} (45)

where (a) follows from that the Euclidean norm of rows of $U$ is unity. For the right-hand side equation in (45) to be satisfied, there are only two possible solutions: $|u_{j',j}|^2 = 0$ or $|u_{j',j}|^2 = 1$, for every $j'$. But it is not possible that all the elements of $U$ are zeros since $\sum_{j'=1}^{N_t} |u_{j',j}|^2 = 1$. Thus, in every column of $U$, there must be only one element whose value is unity, e.g., $\forall j = 1, 2, \ldots, N_t$,

$$|u_{j',j}|^2 = \delta_{j'j_0} \triangleq \begin{cases} 1 & \text{if } j' = j_0, \\ 0 & \text{if } j' \neq j_0, \end{cases}$$  \hspace{1cm} (46)

for some $j_0 \in \{1, 2, \ldots, N_t\}$, where $j_0$ denotes the index of the non-zero element in each column of $U$.

**Achieving the Lower Bound in (43):** From the Cauchy-Schwarz inequality in (44), we obtain an equality if and only if $a$ and $b$ are linearly dependent, i.e., $a = \alpha b$ for some $\alpha \in \mathbb{R}$. Hence, $|u_{i,j}| = \alpha$ for all $i$ and $j$. Since $U$ is unitary, this constraints possible solutions to $|u_{i,j}| = 1/\sqrt{N_t}$.

Now, from (10), it can be observed the $\tilde{m}_j = m_{\max}$, $\forall j = 1, \ldots, r$, if and only if:

$$\sum_{j'=1}^{N_t} |u_{j',j}|^4 = \begin{cases} 1 & \text{if } m \geq 1, \\ 1/N_t & \text{if } 1/2 \leq m < 1, \end{cases}$$  \hspace{1cm} (47)

From the above discussion, $\tilde{m}_j = m_{\max}$, $\forall j = 1, \ldots, r$, if and only if:

$$|u_{j',j}| = \begin{cases} \delta_{j'j_0} & \text{if } m \geq 1, \\ 1/\sqrt{N_t} & \text{if } 1/2 \leq m < 1. \end{cases}$$  \hspace{1cm} (48)
Using a similar argument, and from (10), \( \tilde{m}_j = m_{\min}, \forall j = 1, \ldots, r \), if and only if:

\[
|u_{j,j}'| = \begin{cases} 
1/\sqrt{N_t} & \text{if } m \geq 1, \\
\delta_{j,j} & \text{if } 1/2 \leq m < 1.
\end{cases}
\] (49)

Therefore, from Proposition 1, for a given \( r \), to achieve the maximum \( G_d \), we must have \( m_j = m_{\max}, \forall j = 1, \ldots, r \), i.e., (48) must be satisfied. But, to achieve the maximum \( G_c \), (49) must be satisfied. This gives the properties in Table 1 to achieve the maximum gains and shows the trade-off between \( G_d \) and \( G_c \).

**APPENDIX D**

**PROOF OF PROPOSITION 3**

Using the law of large numbers and the fact that the \( h_{i,j}'s \) are independent, we obtain [36, eq. (10.15)], [27, eq. (2.22)]:

\[
\lim_{N_r \to \infty} \frac{1}{N_r} \mathbf{H}^H \mathbf{H} = \Omega \mathbf{I}_{N_t}.
\] (50)

Hence, we obtain that:

\[
\lim_{N_r \to \infty} \frac{\|\mathbf{D}\|^2_F}{\Omega N_r \sum_{j=1}^{r} \lambda_j} = \lim_{N_r \to \infty} \frac{\text{tr}(\mathbf{H}^H \mathbf{H}(\mathbf{S} - \mathbf{E})(\mathbf{S} - \mathbf{E})^H)}{N_r \text{tr}(\Omega \mathbf{I}_{N_t}(\mathbf{S} - \mathbf{E})(\mathbf{S} - \mathbf{E})^H)}
\] (51)

\[
\overset{(a)}{=} \lim_{N_r \to \infty} \frac{\text{tr}(\mathbf{H}^H \mathbf{H}(\mathbf{S} - \mathbf{E})(\mathbf{S} - \mathbf{E})^H)}{N_r \text{tr}(\lim_{N_r \to \infty} \frac{1}{N_r} \mathbf{H}^H \mathbf{H}(\mathbf{S} - \mathbf{E})(\mathbf{S} - \mathbf{E})^H)}
\] (52)

\[
\overset{(b)}{=} \lim_{N_r \to \infty} \lim_{N_r \to \infty} \frac{\text{tr}(\mathbf{H}^H \mathbf{H}(\mathbf{S} - \mathbf{E})(\mathbf{S} - \mathbf{E})^H)}{N_r \text{tr}(\mathbf{H}^H \mathbf{H}(\mathbf{S} - \mathbf{E})(\mathbf{S} - \mathbf{E})^H)}
\] (53)

\[
\overset{(c)}{=} 1,
\] (54)

where (a) follows from (50); (b) follows from that the continuity of the trace function; and (c) follows from the fact that \( \lim_{N_r \to \infty} \frac{1}{N_r} \text{tr}(\mathbf{H}^H \mathbf{H}(\mathbf{S} - \mathbf{E})(\mathbf{S} - \mathbf{E})^H) \) is finite. Using the result in (54), the following exponential bounds are tight, for large \( N_r \):

\[
\frac{1}{2} \exp \left( -\rho \frac{\|\mathbf{D}\|^2_F}{4} \right) \overset{\Delta}{=} \frac{1}{2} \exp \left( -\rho \frac{\Omega N_r \sum_{j=1}^{r} \lambda_j}{4} \right).
\] (55)

Hence, recalling from Section II that \( \rho = (1/N_t) \log_2(M) E_b/N_0 \), the conditional PEP in (5) as \( N_r \to \infty \) is now bounded as:

\[
\Pr \{ \mathbf{S} \to \mathbf{E} | \mathbf{H} \} \leq \frac{1}{2} \exp \left( -\Omega \frac{\log_2(M) N_r \sum_{j=1}^{r} \lambda_j}{4 N_t} E_b \right).
\] (56)
Now, by taking the expectation over $H$ of both sides of the inequality, we obtain:

$$\Pr\{S \rightarrow E\} \leq \frac{1}{2} \exp\left(-\frac{\Omega \log_2(M) N_r \sum_{j=1}^{r} \lambda_j E_b}{4N_t \frac{N_0}{N_0}}\right)$$

(57)

$$= \frac{1}{2} \prod_{j=1}^{r} \exp\left(\frac{\Omega \log_2(M) E_b}{4N_t \frac{N_0}{N_0} \lambda_j}\right)^{-N_r}.$$  

(58)

APPENDIX E

PROOF OF PROPOSITION 4

We invoke the result in (54). Hence, the PEP in (5), for $N_r/N_t \rightarrow a$ as $N_t, N_r \rightarrow \infty$, can be upper bounded as:

$$\Pr\{S \rightarrow E\} \leq \mathbb{E}_H\left\{\frac{1}{2} \exp\left(-\frac{\rho \|D\|_2^2}{4}\right)\right\}$$

(59)

$$= \frac{1}{2} \exp\left(-\frac{(N_r/N_t) \log_2(M) \Omega \sum_{j=1}^{r} \lambda_j E_b}{4N_t \frac{N_0}{N_0}}\right)$$

(60)

$$= \frac{1}{2} \left(\prod_{j=1}^{r} \exp\left(\frac{\Omega \log_2(M) E_b}{4N_t \frac{N_0}{N_0} \lambda_j}\right)\right)^{-a}. $$

(61)

APPENDIX F

EVALUATING THE PROBABILITY OF BLOCKAGE DUE TO HEIGHTS AND RADIi OF OBSTACLES

We derive expressions for $\Pr\{\xi_1\}$ and $\Pr\{\xi_1\}$, which are the probabilities of blockage events due to the heights and radii of obstacles, respectively. From Fig. 1(a), a blockage event due to the height of an obstacles occurs if $H_0$ is greater than $h(D_0)$, where $H_0$ is the maximum height of the obstacles in the blocking region, i.e., for $n$ obstacles in the blocking region with heights $H_1, H_2, \ldots, H_n$, $H_0 = \max(H_1, H_2, \ldots, H_n)$. Hence, the probability of blockage due to heights of obstacles is given by:

$$\Pr\{\xi_1\} = \Pr\{h_0 < h(D_0)\} = 1 - \mathbb{E}_{D_0}\left\{F_{H_0|D_0}\left(\frac{(h_t - h_r)}{d}D_0 + h_t\right)\right\}$$

(62)

$$= \begin{cases} 0 & h(D_0) \geq h_{\text{max}} \\ \mathbb{E}_{D_0}\left\{\left(\frac{(h_t - h_r)}{d}D_0 + h_t - h_{\text{min}}\right)^n\right\} & h_{\text{min}} \leq h(D_0) < h_{\text{max}} \\ 1 & h(D_0) < h_{\text{min}}. \end{cases}$$

(63)
Now, evaluating (30) for $h_{\min} \leq h(D_0) < h_{\max}$, using the fact that $D_0 \sim U(0, d)$, and using the Binomial Theorem, we obtain:

$$\Pr\{\xi_1\} = 1 - \int_0^d \left\{ \frac{(h_t-h_r)\,d_0 + h_t - h_{\min}}{h_{\max} - h_{\min}} \right\}^n \, dd_0 \quad (64)$$

$$\equiv 1 - \frac{1}{(h_{\max} - h_{\min})^n} \sum_{k=0}^n \binom{n}{k} (h_t - h_{\min})^{n-k} \left( \frac{h_r - h_t}{k+1} \right)^k, \quad (65)$$

where $n$ is the number of obstacles in the blocking region and is equal to $\lceil \lambda_0 dh_tr_{\max} \rceil$ and (a) follows from the binomial expansion of the integrand.

Finding $\Pr\{\xi_2\}$ follows a similar procedure. From Fig. 1(b), the starting point of the proof is $\Pr\{\xi_2\} = \Pr\{R_0 > |y|\}$, where $R_0$ is the radius of the widest obstacle in the blocking region, i.e., $R_0 = \max(R_1, R_2, \ldots, R_n)$. The rest follows as in the previous derivation to obtain

$$\Pr\{\xi_2\} = \mathbb{E}_Y \left\{ \Pr\left\{ R_0 > |y| | Y = y \right\} \right\} \quad (66)$$

$$= 1 - \frac{1}{(r_{\max} - r_{\min})^n} \sum_{k=0}^n \binom{n}{k} (-r_{\min})^{n-k} \frac{r_0^k}{k+1}, \quad (67)$$

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