Space-Time Codes over MIMO
Millimeter-Wave Channels: Design Criteria and New Insights

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Abstract

The error performance limits of millimeter-wave (mm-wave) communication systems are considered. To derive any performance limits of mm-wave communication systems, an accurate statistical channel model must be developed. We argue that the Nakagami-$m$ distribution accurately models the statistics of small-scale fading at mm-wave frequencies and provide extensive numerical simulations for the indoor environment at 60 GHz. Then, we derive an upper bound on the error performance of any space-time coding scheme and show the improvements due to communicating over the millimeter wave channel, in terms of diversity and coding gains. Our upper bound is based on the pairwise error probability (PEP). We then examine the resulting diversity and coding gains to propose design criteria that maximize the diversity and coding gains. Orthogonal space-time block codes are shown to achieve the maximum diversity gain but not the maximum coding gain. Indeed, we show that there exists a trade-off between the diversity gain and the coding gain. Furthermore, we investigate the effect of blockage on the error performance using stochastic geometry. Our analysis and simulations show that blockage only reduces the coding gain and does not affect the diversity gain. This reduction in the coding gain is a function of the probability of line-of-sight (LOS) communication, power loss exponents, the distance between transceivers, and the coding gain without considering the effect of blockage. For instance, in a typical indoor environment for mm-wave communication, blockage due to humans or other obstacles can reduce the coding gain by up to 3 dB for a bit error probability of $10^{-3}$.

Keywords

MIMO, mm-wave, wireless channel modeling, pairwise error probability, space-time coding, Nakagami-$m$, diversity gain, coding gain, blockage, stochastic geometry.

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I. INTRODUCTION

A. Motivation

Communicating via the 5G standard is expected to occur also at wireless mm-wave frequencies (30 GHz – 100 GHz), as opposed to only communicating over the 700 MHz – 2.6 GHz bandwidth in previous and current standards [1]. This massive expansion in bandwidth promises alleviating spectrum scarcity and providing high data rates to users [1]–[3]. Further, the Federal Communications Commission (FCC) publicly recognizes that communicating over the mm-wave spectrum could support an increased number of users through spectrum reuse – which is possible due to the directivity of the channel and the high frequency communication resulting in reduced-size, smart antenna arrays [4]. Thus, this high potential for the mm-wave spectrum to revolutionize cellular communication is why the community is highly interested in modeling and understanding the performance limits of the mm-wave channel, such as [1]–[14].

B. Literature Review

Reliable communication over the mm-wave channel requires accurate models of the wireless channel. These models may differ from conventional radio frequency (RF) channel models. Furthermore, statistical models of the channel can be used to analyze performance metrics such coverage probability, outage probability, and error probability, as in [5], [6], [9], [10]. In [5] and [6], the authors analyze the coverage probability for mm-wave cellular networks assuming the channel coefficients that model small-scale fading are Nakagami-$m$-distributed. But in [9] and [10], the error and outage performances of mm-wave relaying channels are analyzed assuming Rayleigh and Rician small-scale fading coefficients, respectively. Also, in [8], a space-time coding scheme is proposed to improve the error performance over mm-wave channels assuming the Rician fading channel model. Moreover, the seminal works by Tarokh et al. [15], [16] analyze the error performance of space-time codes over Rayleigh and Rician fading channels. However, the Nakagami-$m$ distribution can more accurately model small-scale fading over MIMO mm-wave channels [5], [6], [17, pp. 78–79].

In [12], the authors propose a statistical method for modeling wireless channels at 60 GHz. This statistical model, however, is too computationally demanding since it computes probability distribution functions (pdfs) of virtual sources’ (VSs) locations before using Monte Carlo (MC)
simulations to calculate the channel coefficients. On the other hand, the authors in [14] propose a more computationally efficient method for modeling the mm-wave wireless channel than the method proposed in [12]. The authors in [14] published their method in an open source software program tailored as mm-Trace. While the proposed model in [14] allows for investigating interference at mm-wave frequencies when having multiple transceivers, their proposed model is only two-dimensional. A model for the mm-wave channel needs to be three-dimensional to be useful for applications involving smart antennas [12]. In [7], an experimental investigation for indoor channel at 60 GHz for a conference room is proposed, which contrast with the geometry-based approaches (that use environment properties and ray-tracing rather that measurements) in [12], [14]. We consider modeling techniques to find a closed-form expression of the statistics of small scale fading.

Closed-form expressions for the statistics of the wireless channel also allow for accurate and intuitive assessment of its performance limits [15]. In addition, analytical expressions for the statistics of the channel allow for establishing clear design criteria for space-time codes to achieve the maximum diversity and coding gains of the channel. In the seminal work of Tarokh et al. [15], design criteria for multiple-input-multiple-output (MIMO) Rician and Rayleigh fading channels were proposed from derivations of the average pairwise error probability (PEP). Although the PEP for single-input-single-output (SISO) and single-input-multiple-output (SIMO) systems under Nakagami-\(m\) fading channels is known and can be easily derived, efforts to characterize the average PEP over Nakagami-\(m\) fading channels for MIMO communication systems have been less successful. The reason, as will be discussed in detail in subsection III-B, is the lack of a tractable, closed-form expressions for the exact distribution of the amplitude of the sum of complex-valued, Nakagami-\(m\)-distributed random variables. In [18] and [19], exact expressions for the PEP over Nakagami-\(m\) fading MIMO channels are proposed in terms of an infinite series and integral forms, respectively. These exact expressions, however, preclude the deduction of diversity and coding gains and inference of design criteria as performed in [15] for Rayleigh and Rician fading channels.

Finally, the known directivity of the mm-wave channel makes blockage an even more significant phenomenon that needs incorporating into any model and analysis of the wireless channel [2], [3], [5], [6], [20], [21]. In [20], a framework for analyzing performance metrics for mm-wave cellular networks using stochastic geometry is proposed. In [5] and [6], objects that can
cause blockage are modeled using a Poisson point process (PPP) with a certain density that is considered to be a system parameter. Using stochastic geometry, the Thinning Theorem can then be invoked [22] to obtain effective densities of obstacles that can cause blockage (also known as densities of obstacles in the blocking region). For instance, in [5] and [6], this method is used to derive and analyze outage and coverage probabilities for mm-wave cellular networks. In [21], stochastic geometry is used to analyze the capacity of mm-wave networks. In [23], the authors derive the rate and coverage probability for outdoor cellular networks using a ball-based blockage model.

C. Contribution

In this paper, which is a substantial enhancement of [24], we derive design criteria for space-time codes used over the mm-wave channel. Our contributions can be summarized as follows.

1) We modify the method in [12] to model the statistics of small-scale fading at 60 GHz for the indoor channel in a computationally efficient way. That is, we sample from assumed distributions of transceivers’ locations; compute locations of prominent virtual sources for each sample using ray-tracing; and then compute the empirical channel transfer function (CTF). We identify the Nakagami-\(m\) cumulative distribution function (cdf) to closely match the obtained empirical cumulative distribution function (ecdf) of the CTF. Our approach results in a model that is three-dimensional unlike the model in [14]. We also compare the statistics of small-scale fading obtained using our approach with the measurements-based, statistical method in [7], thus, confirming that the Nakagami-\(m\) distribution is a good statistical channel model for indoor mm-wave communication systems.

2) To gain insights into fundamental error-performance limits of mm-wave channels, we derive an upper bound on the error performance for any space-time code over Nakagami-\(m\) fading MIMO channels; thus, characterizing the diversity and coding gains. Our upper bound is based on the PEP. To this end, we use a result by Nakagami in [25] to obtain a closed-form expression for the sum of complex Nakagami-\(m\)-distributed random variables. We then infer design criteria for space-time codes to maximize the achievable diversity gain from the derived PEP expression. We deduce a property of space-time codes that result in achieving the maximum possible diversity gain under Nakagami-\(m\) fading
channels. Further, we show that the determinant criterion for space-time codes in [15] does not necessarily lead to the maximum coding gain. But, for space-time codes with a certain symmetrical property, the well-known determinant criterion shows up. Indeed, we show that there is a trade-off between the diversity and coding gains for space-time codes communicating over Nakagami-\(m\) fading channels.

3) We use stochastic geometry to study the effect of blockage, due to objects in indoor environments, on the error performance of mm-wave communication systems. We show that the diversity gain is unaffected by blockage but the coding gain in presence of blockage is reduced, with the reduction being a function of the probability of line-of-sight (LOS) communication, power loss exponents, the distance between transceivers, and the coding gain without considering the effect of blockage.

Note that research works such as [18], [19], [26] analyze the PEP for MIMO, Nakagami-\(m\) fading channels but provide integral-forms of the average PEP over the channel fading, which precludes inference of design criteria as in [15] for Rician and Rayleigh fading channels. To the best of the authors’ knowledge, there has been no previous works on deriving design criteria for space-time codes communicating over MIMO, Nakagami-\(m\) fading channels (contribution 2). Furthermore, Previous studies, such as [5], [6], [9], [10], [20], [21], [23], have used stochastic geometry to derive the coverage probability and capacity for mm-wave channels. But, to the best of the authors’ knowledge, there has been no research on the error performance of space-time codes over the mm-wave channel that considers blockage (contribution 3).

D. Outline

The remainder of this paper is outlined as follows. In section II, we briefly review the modeling of multipath fading in indoor environments, and show that the statistics of multipath fading follow a Nakagami-\(m\) distribution. In section III, we derive the PEP for Nakagami-\(m\) fading, MIMO channels with channel state information at the receiver (CSI-R). We use our result to infer design criteria for space-time codes of Nakagami-\(m\) fading channels in subsection III-D. In section IV, we consider the effect of blockage in MIMO systems on the PEP. In section V, we simulate several space-time block codes for multiple transceivers communication systems, illustrating our design criteria. Finally, in section VI, we summarize our main results and point out new research directions.
E. Notation

We represent an arbitrary $N \times M$ matrix $A$ with complex entries $a_{i,j}$ by $A = [a_{i,j}] \in \mathbb{C}^{N \times M}$. The square of the Frobenius norm of an arbitrary matrix $A = [a_{i,j}] \in \mathbb{C}^{N \times M}$ is defined as $\|A\|^2_F = \text{tr}(A^H A) = \sum_{i=1}^{N} \sum_{j=1}^{M} |a_{i,j}|^2$, where $\text{tr}(A)$ represents the trace of $A$, and $A^H$ denotes the Hermitian of $A$. We denote the Euclidean norm of a real-valued vector $r \in \mathbb{R}^{N \times 1}$ by $\|r\|_2$.

$\text{Pr}\{\zeta\}$ denotes the probability measure over some sample space $\Omega$ of a subset $\zeta \subset \Omega$. We represent a random variable by an uppercase letter (e.g. $Y$) and its realization by a lowercase letter (e.g. $y$), i.e., the pdf of $Y$ is $f_Y(y)$. $\mathbb{E}_Y\{\cdot\}$ denotes the expectation of the expression inside the braces over random variable $Y$; we omit the subscript when the expectation is taken over all random variables inside the braces. We represent a random variable $X$ that is Nakagami-$m$-distributed with parameters $m$ and $\Omega$ by $X \sim \text{Nakagami}(m, \Omega)$ (where $m \geq 1/2$ and $\Omega > 0$), and a random variable $Y$ that is uniformly distributed over an interval $(a,b) \subset \mathbb{R}$ by $X \sim \text{U}(a,b)$. We denote the amplitude and phase of a complex number $h \in \mathbb{C}$ by $|h|$ and $\angle h$, respectively.

We use the equal sign $\equiv$ as in $f(x) \equiv g(x)$ to denote that $\lim_{x \to \infty} f(x)/g(x) = 1$.

II. Indoor Channel Models of Multipath Fading

We are interested in modeling the statistics of multipath fading in mm-wave bands. When different multipath components arrive at the receiver, they produce a random pattern of constructive and destructive interference called multipath fading [17]. Multipath fading changes fast enough to cause bit errors during communication over time-scales that are short enough so they cannot be dealt with efficiently at a protocol level. Hence, as in [15], [16], [27], [28], only the statistic of small-scale fading are used for the error analysis of space-time codes. Modeling power loss and shadowing over the mm-wave channel has been extensively studied in [2] and the references therein and can be used for network-level analysis.

The author [12] models small-scale fading for the mm-wave channel using ray-tracing and pdfs that describe the propagation paths and amplitudes of received waves. This geometry-based approach for modeling channels contrasts with measurements-based models. Measurements-based models assume the channel is random enough in a sense that inferring the channel statistics from measurements provides a good model for the channel in general [12]. But the mm-wave channel lacks this randomness due to the high loss in energy experienced by traveling waves resulting from the high (GHz) frequency propagation. As a result, the mm-wave channel model
depends mainly on environmental properties. Furthermore, the pdf of multipath amplitudes can be computed from Friis’s transmission equation, given the pdf of locations of VSs and environmental properties (such as room geometry, dielectric constants of wall material, signaling frequency, etc.). The properties of the environment are assumed to be known and the distribution of VSs $R_k$ is computed from distributions of transceivers’ locations using the law of total probability. Let $f(r_k)$ denote the pdf of the random variable $R_k$. Then, using the law of total probability,

$$f(r_k) = \int_A \int_{R_{Rx}} \int_{R_{Tx}} f(r_k | r_{TX}, r_{RX}, a) f(r_{TX}) f(r_{RX}) f(a) d r_{TX} d r_{RX} d a,$$

where $f(r_{TX})$ and $f(r_{RX})$ are the pdfs of the transmitter and receiver locations, respectively; $p(a)$ is the pdf of a three-dimensional vector $A$ whose components are uniformly distributed over the whole three-dimensional room; and $f(r_k | r_{TX}, r_{RX}, a)$ is the conditional pdf of having a virtual source at location $r_k$ given the transmitter and receiver are located at positions $r_{Tx}$ and $r_{Rx}$, respectively, and given $a$ [12].

The main disadvantage of modeling the mm-wave channel using the modeling approach in [12] is the complexity involved in computing (1). Note that (1) is an integral over three variables each being three dimensional. So, it is equivalent to taking nine integrals. The time required to compute integrals using most numerical integration methods grows exponentially with the number of integrals that needs to be evaluated [29]. After finding the distribution of $r_k$, we would still have to use Friis’s transmission equation to relate the distance traveled by waves and the geometrical properties of an environment with the channel response. Then, we would need to find the pdf of the CTF as a function of random variable $r_k$. To ameliorate the computational complexity, we use MC simulations to directly obtain the statistics of the CTF.

A. The CTF from Geometry and Properties of The Environment

First, the pdfs of the transceivers’ locations are specified as in [12]. Then, rather than evaluating (1) which is computationally prohibitive, we sample locations of transceivers from their assumed distributions. Samples of transceivers’ locations along with the room geometry dictate the positions of VSs. From simple geometry, ray-tracing can compute the distance traveled by waves as they propagate from the transmitter to the receiver. This distance, along with environmental properties, can then be used to find the CTF. For $M$ rays traveling from transmitter to receiver, the transfer function $H(f)$ is sum of the contribution of each ray. Let $R_i$ be a random three-dimensional vector containing the distance traveled along each axis from transmitter to the
receiver. Then, \( H(f) \) at the receiver is given by:

\[
H(f) = \sum_{i=1}^{M} \frac{\lambda \sqrt{G_{Tx} G_{Rx}} C_{Tx}(\theta, \psi) C_{Rx}(\theta, \psi)}{4\pi \|R_i\|_2} \prod_{j=1}^{l_{max}} F(\theta_{i,j}) C_{Tx}(\theta, \psi) \exp\left(-j \frac{2\pi}{\lambda} \|R_i\|_2\right),
\]

(2)

where \( \lambda = \frac{f c}{c} \) is the wavelength, \( f \) is the signaling frequency, \( c \) is the speed of light; \( G_{Tx} \) and \( G_{Rx} \) are the antennas’ gains at the transmitter and receiver, respectively; \( C_{Tx} \) and \( C_{Rx} \) are the radiation patterns at the transmitter and receiver, respectively; \( F(\theta_{i,j}) \) is Fresnel’s reflection coefficient; and \( l_{max} \) is the number of the maximum possible reflected rays \[12\]. Evaluating the CTF in (2) requires finding the distance \( \|R_i\|_2 \) traveled by each wave. Here, instead of computing the pdf of \( \|R_i\|_2 \), we use ray-tracing to find locations of virtual sources. VSSs are virtual transmitters of reflected rays arriving at the receiver. So, the distance traveled by a wave can be computed as the distance traveled from the source to the VSS’s location plus distance traveled from the VSS’s location to the receiver. To account for amplitude attenuation and phase shift due to reflections off walls of a room, we compute Fresnel’s reflection coefficients for each ray. Then, using parameters of the environment and \( \|R_i\|_2 \), we compute a sample of \( H(f) \) from (2). The process then repeats until enough samples are obtained to generate an empirical distribution. Algorithm 1 summarizes the procedure for generating the statistics of \( H(f) \).

**B. Simulation of the Statistics of Multipath Fading for Indoor Environments**

We implemented our proposed algorithm (Algorithm 1) to find the CTF for a room of dimensions \( 4 \text{ m} \times 4 \text{ m} \times 4 \text{ m} \) at a frequency of 60 GHz as in \[12\], \[13\]. For simplicity and since we are interested here in studying the statistics of the channel and not the effect of antenna gains, the antennas’ gains \( G_{Tx} \) and \( G_{Rx} \) were assumed to be unity. The transceivers’ locations statistics represented by \( f(r_{Tx}) \) and \( f(r_{Rx}) \) are uniformly distributed over the ceiling and over the volume between one fifth and one half the height of the room, respectively. These assumption are justified in \[12\]. Then, samples from the transmitter’s and receiver’s locations’ distributions were used to ray-trace locations of virtual sources to compute \( \|R_i\| \). Further, for each reflected ray in the room, Fresnel’s reflection coefficients were computed at each wall in the room, as is described in more detail using a ray tracing tree in \[30\]. The room geometry and dielectric constant of walls were acquired from \[12\], \[13\]. \( H(f) \) was computed for each wave reaching the receiver using (2). For each order of reflected waves, five rays might possibly reach the receiver: one due to the line of sight component, and the remaining four rays are due
Algorithm 1  Proposed MC-Based Method for Obtaining the Statistics of $H(f)$

1: Assume distributions of transceivers’ and obstacles’ locations.
2: Sample from these distributions.
3: for each sample of transceivers locations do
4: Compute locations of virtual sources for each wall using ray-tracing described above.
5: for each wall in the room do
6: Compute angle of incidence $\theta$.
7: Compute Fresnel’s reflection coefficients.
8: end for
9: Sum contributions to $H(f)$ from each ray as in equation (2).
10: end for
11: Repeat steps 2 to 9 $N$ times.
12: Compute empirical cdf of $|H(f)|$ and $\angle H(f)$.

This gives us the statistics of $H(f)$.

TABLE I: Shows the parameters of distributions used to fit cdf of $|H(f)|$ and the mean squared error for between each cdf fit and the empirical cdf.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Parameters [31]</th>
<th>Mean Squared Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gamma</td>
<td>$\alpha = 13.5088, \beta = 0.0740$</td>
<td>$3.6006 \times 10^{-5}$</td>
</tr>
<tr>
<td>Lognormal</td>
<td>$\alpha = -0.0375, \beta = 0.2799$</td>
<td>$2.0427 \times 10^{-4}$</td>
</tr>
<tr>
<td>Nakagami-$m$</td>
<td>$m = 3.6350, \Omega = 1.0713$</td>
<td>$1.6757 \times 10^{-5}$</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>$\beta = 0.7319$</td>
<td>$0.0158$</td>
</tr>
<tr>
<td>Rician</td>
<td>$\alpha = 0.9602, \beta = 0.2732$</td>
<td>$1.1934 \times 10^{-4}$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$\alpha = -3.1395, \beta = 3.1401$</td>
<td>$2.0427 \times 10^{-4}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\mu = 1.1001, \Omega = 0.2105$</td>
<td>$3.1668 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

to reflections off walls in the room. The assumption that only such a small number of rays is able to reach the receiver is due to the high power loss experienced by propagating waves due to the high propagation frequency [7], [12].

Figure 1 compares ecdfs of the amplitude and phase of the CTF obtained from Algorithm 1 and from the statistical model in [7] with common maximum likelihood (ML) cdf fits. In contrast
with our simulated geometry-based approach, the statistical model in [7] is motivated by their experimental investigation of a conference room at 60 GHz. We are interested in finding the statistics of small-scale fading at 60 GHz for indoor environments. Figures 1(a) and 1(b) show that Nakagami-$m$ and Gamma distributions are accurate fits for the probability distribution of the amplitude $|H(f)|$ of the CTF. This is true for both our proposed model and the statistical model in [7]. That is, for the indoor environment we considered here and in [7], the Nakagami-$m$ distribution is a good fit for the statistics of the CTF. Figure 1(c) shows that the Rayleigh distribution is not a good fit for the statistics of $|H(f)|$ according to both models. Figure 1(d) shows that the a uniform distribution between $-\pi$ and $\pi$ accurately describes the statistics of
the phase $\angle H(f)$ of the CTF.

We computed the mean square error (MSE) to measure how accurately a probability distribution matches the ecdf of simulated amplitudes and phases of the CTF. Table I summarizes MSE between empirical and analytic cdfs. The definitions of the distributions in Table I can be found in [31]. We find that the best cdf fits for the obtained ecdf of the amplitude are the Gamma and Nakagami-$m$ cdfs since their maximum likelihood fits have the least MSEs, which is in order of $10^{-5}$. We also observe that the distribution of phase of the CTF is approximately uniform between $-\pi$ and $\pi$. This observed uniformity of the distribution of the phase of the CTF satisfies a conjecture in the Saleh-Valenzuela model [32] which states that the phase is uniformly distributed regardless of the signaling frequency for indoor environments. Thus, we can accurately model the multipath fading statistics of the amplitude $|H(f)|$ and phase $\angle H(f)$ of the CTF in the indoor environment specified using a Nakagami-$m$ distribution and a uniform distribution, respectively.

We showed that the amplitude and phase of multipath fading of the CTF at the mm-wave spectrum (at 60 GHz) in an indoor environment are accurately modeled by a Nakagami-$m$ distribution and a uniform distribution, respectively. In the next section, we use this model to derive a general, closed-form expression for a tight upper bound on the pairwise error probability of space-time codes. Such an expression provides more insights (in terms of diversity and coding gains) into the error performance of space-time codes over mm-wave communication systems than numerical simulations alone.

III. PERFORMANCE ANALYSIS

A. System Model and Related Background

The mm-wave channel is known to be wideband but we assume a flat fading channel since any wideband channel can be split into a set of parallel narrowband channels using orthogonalization techniques [17]. Under the narrowband assumption, the realizations of the CTF are constant over all frequencies such that frequency components of transmitted signals experience the same multiplicative, complex gain. As a result, the convolution of the time-domain representation of the transmitted signal and the channel impulse response simplifies to a multiplication of the transmitted signal by a constant. We also assume the realization of the channel matrix is constant over the length of the code block (i.e., quasi-static fading), and the receiver has perfect channel
state information (CSI-R). These assumptions are also justified and discussed in [5], [6], [15], [16], [27], [28].

For a MIMO system with $N_t$ and $N_r$ transmit and receive antennas, respectively, and considering a block code of length $T$, the received symbols matrix $\mathbf{R} = [r_{i,j}] \in \mathbb{C}^{N_r \times T}$ is given by

$$\mathbf{R} = \sqrt{\rho} \mathbf{H} \mathbf{S} + \mathbf{Z},$$

(3)

where $\mathbf{H} = [h_{i,j}] \in \mathbb{C}^{N_r \times N_t}$ is the normalized channel matrix; $\mathbf{S} = [s_{i,j}] \in \mathbb{C}^{N_t \times T}$ is the normalized transmitted symbols matrix; $\mathbf{Z} = [z_{i,j}] \in \mathbb{C}^{N_r \times T}$ is the normalized AWGN matrix such that $z_{i,j} \sim \mathcal{CN}(0, 1)$; and the average SNR at the transmitter is denoted by $\rho \in [0, \infty)$. Each transmitted symbol $s_{i,j}$ belongs to the set of allowable alphabets $\mathcal{M}$, i.e., $s_{i,j} \in \mathcal{M}$, where $M = |\mathcal{M}|$. We define $P_b$ as the bit error probability of the channel in (3). In terms of the diversity gain $G_d$ and the coding gain $G_c$, $P_b$ can be expressed as

$$P_b = k \left( G_c \frac{E_b}{N_0} \right)^{-G_d},$$

(4)

where $E_b/N_0$ is the bit energy to noise ratio and $k$ is some real-valued, positive constant [28, eq. 3.19], [27, eq. (1.3)]. Unfortunately, it is difficult to compute $P_b$ in a closed-form. However, at high SNR, a closed-form expression of the bit error probability can be obtained from an average pairwise error probability (PEP). The PEP is the probability of obtaining codeword $\mathbf{E} = [e_{i,j}] \in \mathbb{C}^{N_r \times T}$ at the receiver when codeword $\mathbf{S}$ was transmitted under channel conditions $\mathbf{H}$, at SNR $\rho$. Indeed, the average PEP $\Pr\{\mathbf{S} \rightarrow \mathbf{E}\}$ is itself an upper bound on the symbol error probability $P_s$, see [27, pp. 136-137]. But, as $\rho \rightarrow \infty$, the two error probabilities converge to the same value times a constant $c > 0$ [27, pp. 142-143], i.e., $P_s \doteq c \Pr\{\mathbf{S} \rightarrow \mathbf{E}\}$. Furthermore, for a codeword of length $T$ and a modulation order of $M$, $P_b$ is related to $P_s$ as follows [27]:

$$P_b \doteq \frac{P_s}{T \log_2 M}.$$  

(5)

Hence, in the high SNR regime, $P_b \doteq \frac{c}{T \log_2 M} \Pr\{\mathbf{S} \rightarrow \mathbf{E}\}$, and we can use closed-form expressions of the PEP (cf. III-C) to obtain diversity and coding gains from (4). Now, assuming $\mathbf{R}$ is decoded at the receiver using a maximum likelihood decoding rule, it can be easily shown that the conditional PEP on the channel matrix is equal to [15], [16], [27]:

$$\Pr\{\mathbf{S} \rightarrow \mathbf{E}|\mathbf{H}\} = Q\left( \sqrt{\frac{\rho}{2}} \|\mathbf{D}\|_F \right),$$

(6)
where \( D \triangleq H(S - E) \), \( Q(\cdot) \) is the Q-function, and \( \|D\|_F \) is the Frobenius norm of \( D \). Using the Chernoff bound on the Q-function, the conditional PEP can be upper bounded as [27]:

\[
\Pr\{S \to E|H\} \leq \frac{1}{2} \exp\left(-\rho \frac{\|D\|_F^2}{4}\right).
\] (7)

It can be easily shown using simple algebraic manipulations that \( \|D\|_F^2 \) can be written as [27, pp. 138–140], [15]:

\[
\|D\|_F^2 = \sum_{i=1}^{N_r} \sum_{j=1}^{r} \lambda_j |\beta_{i,j}|^2,
\] (8)

where \( \|D\|_F^2 = \text{tr}\{(S - E)(S - E)^HH^H\} \), \( \{\lambda_j\}_{j=1}^r \) and \( r \) are the non-zero eigenvalues and rank of \( (S - E)(S - E)^H \), respectively, and \( \beta_{i,j} \) is defined as

\[
\beta_{i,j} \triangleq \sum_{j'=1}^{N_t} u_{i,j'} h_{i,j'},
\] (9)

where \( U = [u_{i,j}] \in \mathbb{C}^{N_t \times N_t} \) is unitary and orthogonal (i.e., \( UU^H = I \)), resulting from the singular value decomposition \( (S - E)(S - E)^H = U\Lambda U^H \).

Hence, by substituting (8) into (7), an upper bound on the conditional PEP can be written as:

\[
\Pr\{S \to E|H\} \leq \frac{1}{2} \exp\left(-\rho \frac{\sum_{i=1}^{N_r} \sum_{j=1}^{r} \lambda_j |\beta_{i,j}|^2}{4}\right)
= \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \exp\left(-\frac{\rho}{4} \lambda_j |\beta_{i,j}|^2\right).
\] (10)

To get a closed-form expression for an upper bound on PEP under Nakagami-\( m \) fading, we need to average (10) over the channel coefficients. But diversity and coding gains derived from an upper bound on the PEP might not themselves constitute exact diversity and coding gains. Fortunately, as we show in the following lemma, the exponential bound in (10) is tight at high SNR.

**Lemma 1:** \( Q(x) \) and \( \frac{1}{2} \exp(-\frac{1}{2}x^2) \) converge at the same rate and to the same value in the limit as \( x \) goes to \( \infty \), i.e., \( Q(x) \approx \frac{1}{2} \exp(-\frac{1}{2}x^2) \).

**Proof:** From [33, Theorem 1], \( \alpha_1 \exp(-\beta_1 x^2) \) is an upper bound on \( Q(x) \) iff \( \alpha_1 \geq 1/2 \) and \( 0 \leq \beta_1 \leq 1/2 \). And, from [33, Theorem 2], \( \alpha_2 \exp(-\beta_2 x^2) \) is a lower bound on \( Q(x) \) if and
only if $\beta_2 > 1/2$ and $0 \leq \alpha_2 \leq \sqrt{\frac{e^{\beta_2-1}}{2\pi \beta_2^2}}$. Taking the limit as $x$ goes to $\infty$ in $Q(x)$ and its exponential bounds, and dividing by $k_1 \exp(-\frac{1}{2} x^2)$, it hence follows that:

$$\frac{\alpha_2 \exp(-\beta_2 x^2)}{\frac{1}{2} \exp(-\frac{1}{2} x^2)} \leq \frac{Q(x)}{\frac{1}{2} \exp(-\frac{1}{2} x^2)} \leq \frac{\alpha_1 \exp(-\beta_1 x^2)}{\frac{1}{2} \exp(-\frac{1}{2} x^2)}. \quad (11)$$

Now, in (11), we set $\beta_1 = 1/2$, and take the limit as $\beta_2$ and $x$ go to $1/2$ and $\infty$, respectively. Thus, both the lower and the upper bounds become equal to $2\alpha_2$ and $2\alpha_1$, respectively. Now, it is permissible to set $\alpha_1 = \alpha_2 = 1/2$; making both lower and upper bounds equal to $1$. Therefore, we have that:

$$\lim_{x \to \infty} \left\{ \frac{Q(x)}{\frac{1}{2} \exp(-\frac{1}{2} x^2)} \right\} = 1. \quad (12)$$

Hence, Lemma 1 allows inferring actual diversity and coding gains from (10), rather than only bounds on these gains.

B. Statistics of the Envelope of a Linear Combination of Complex Nakagami-$m$ Variables

We use equation (10) to derive an upper bound on PEP under Nakagami-$m$ fading. Using the model of the statistics of the channel coefficients justified, we have that $|h_{i,j}| \sim \text{Nakagami}(m, \Omega)$ and $\angle h_{i,j} \sim U(-\pi, \pi)$. To average the expression in (10), we first need to find the pdf of $|\beta_{i,j}| = |\sum_{j'=1}^{N_t} u_{j',j} h_{i,j'}|$. Note that $|\beta_{i,j}|$ can be viewed as the envelope of the sum of $N_t$ weighted complex random variables, where the weights have a unity Euclidean norm. Each random variable in this sum has a Nakagami-$m$-distributed amplitude and a uniformly distributed phase. An exact, general integral-form for the pdf of the magnitude of the sum of independent, complex Nakagami-$m$ random variables with phases that are statistically independent from amplitudes was first proposed by Nakagami [25]. Later, Du et al. [34] derived this integral-form. Nonetheless, Nakagami [25] and Du et al. [34] also established that the pdf of the envelope of this sum of Nakagami-$m$-distributed random variables is well-approximated by another Nakagami-$m$ distribution with parameters $\tilde{m}$ and $\tilde{\Omega}$. Note that the problem we are considering here is the sum of complex Nakagami-$m$-distributed random variable; hence, the known pdf for the sum of real Nakagami-$m$ random variables proposed in [35], which is widely used for performance analysis of communication systems, cannot be used.

To obtain an intuitive understanding from an upper limit on the PEP in terms of the diversity gain for MIMO systems under Nakagami-$m$ fading, we approximate the pdf of $|\beta_{i,j}|$
using a Nakagami-$m$ distribution with parameters $\tilde{m}$ and $\tilde{\Omega}$ as defined in [34]. For $|h_{i,j}| \sim$ Nakagami($m, \Omega$), $|\beta_{i,j}|$ is the envelope of the weighted sum of independent, complex Nakagami-$m$ random variables, such that each term in the summation has an amplitude of $|u_{i,j}| |h_{i,j}| \sim$ Nakagami($|u_{i,j}|^2 m, \Omega$) and a phase of ($\angle h_{i,j} + \angle u_{i,j}$) \sim U(-\pi + \angle u_{i,j}, \pi + \angle u_{i,j})$. Therefore, by applying [34, eq. (17)], we have that $|\beta_{i,j}| \sim$ Nakagami($\tilde{m}, \tilde{\Omega}$) where

$$\tilde{\Omega} = \sum_{j'=1}^{N_t} |u_{j',j}|^2 \Omega = \Omega,$$

$$\tilde{m}_j = \frac{\sum_{j'=1}^{N_t} |u_{j',j}|^4 + m \sum_{j'=1}^{N_t} \sum_{j''=1}^{N_t} |u_{j',j}|^2 |u_{j'',j}|^2}{\sum_{j'=1}^{N_t} |u_{j',j}|^4}.$$ (14)

Note that, for the case when $m = 1$ (i.e., Rayleigh fading), the denominator of (14) can be written as $\sum_{j'=1}^{N_t} |u_{j',j}|^2 \sum_{j''=1}^{N_t} |u_{j'',j}|^2 = 1$ since matrix $U$ is unitary. Hence, $\tilde{m}_j$ in (14) is equal to 1, i.e., the pdf of $|\beta_{i,j}|$ is Rayleigh, in agreement with [34].

The expression for $\tilde{m}_j$ in (14) depends on the codeword choice through elements of matrix $U$. To find bounds on $\tilde{m}_j$ that are independent of the codeword and are useful for inferring design criteria, we add and subtract the term for which $j'' = j'$ for the expression of the denominator in (14), which is given by:

$$\sum_{j'=1}^{N_t} |u_{j',j}|^4 + m \left( \sum_{j'=1}^{N_t} \sum_{j''=1}^{N_t} |u_{j',j}|^2 |u_{j'',j}|^2 - \sum_{j'=1}^{N_t} |u_{j',j}|^4 \right) = (1 - m) \sum_{j'=1}^{N_t} |u_{j',j}|^4 + m.$$ (15)

To bound the expression in (15) we bound $\sum_{j'=1}^{N_t} |u_{j',j}|^4$: $\forall u_{j',j} \in \mathbb{C}$: $\sum_{j'=1}^{N_t} |u_{j',j}|^2 = 1$, (using the Cauchy-Schwarz inequality to obtain the non-zero lower bound)

$$0 < \frac{1}{N_t} \leq \sum_{j'=1}^{N_t} |u_{j',j}|^4 \leq \sum_{j'=1}^{N_t} |u_{j',j}|^2 = 1.$$ (16)

Hence, from the inequality in (16), the expression in (15) is bounded in $(m, 1]$ for $1/2 \leq m < 1$, and (15) is bounded in $[1, m)$ for $m \geq 1$. Therefore, by substituting these bound in the expression for $\tilde{m}_j$ in (14), $\tilde{m}_j$ can be bounded according to:

$$\min(1, m) < \tilde{m}_j \leq \max(1, m) \quad \text{for } m \geq 1/2.$$ (17)

We will find the bounds in (17) useful for deriving design criteria in subsection III-D.
C. Upper Bound on PEP under Nakagami-m Fading for MIMO Systems

Now that we know the pdf of \( \beta_{i,j} \) and we bounded \( \tilde{m}_j \) with bounds that are independent of the codeword choice, we average the PEP over the channel statistics to obtain the diversity and coding gains. Averaging over \(|\beta_{i,j}|\) in (10), and for \( h_{i,j}'s \) independent and identically distributed (i.i.d.), we obtain

\[
\Pr\{ S \to E \} \leq \mathbb{E} \left\{ \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \exp \left( -\frac{\rho}{4} \lambda_j |\beta_{i,j}|^2 \right) \right\}
\]

\[
= \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \mathbb{E} \left\{ \exp \left( -\frac{\rho}{4} \lambda_j |\beta_{i,j}|^2 \right) \right\}
\]

\[
= \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \int_0^\infty \exp \left( -\frac{\rho}{4} \lambda_j x^2 \right) \frac{2\tilde{m}_j}{\Gamma(\tilde{m}_j)\Omega^{\tilde{m}_j}} x^{2\tilde{m}_j-1} \exp \left( -\frac{\tilde{m}_j x^2}{\Omega} \right) dx
\]

\[
= \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \tilde{m}_j \Omega^{\tilde{m}_j} \left( (\rho/4)\lambda_j + \tilde{m}_j/\Omega \right)^{\tilde{m}_j},
\]

where (a) follows from that \( \beta_{i,j}'s \) are functions of statistically independent random variables and are, hence, also independent (but not necessarily identically distributed); (b) follows from applying the definition of the pdf of \(|\beta_{i,j}|\), \( f_{|\beta_{i,j}|}(x) \); and (c) follows from that \( \int_0^\infty x^n \exp(-ax^2)dx = \Gamma(n+1)/(2a^{n+1/2}) \), for \( a > 0 \) and \( n > -1 \) [36].

Now, in the high SNR regime, \((\rho/4)\lambda_j \gg \tilde{m}_j/\Omega\), and by invoking Lemma 1, we can write the PEP as:

\[
\Pr\{ S \to E \} \leq \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \tilde{m}_j \Omega^{\tilde{m}_j} \left( (\rho/4)\lambda_j + \tilde{m}_j/\Omega \right)^{\tilde{m}_j}
\]

\[
= \frac{1}{2} \left( \prod_{j=1}^{r} \left( \frac{\lambda_j}{\tilde{m}_j} \sum_{j=1}^{N_r} \tilde{m}_j \right) \frac{\tilde{\Omega}}{\rho} \right)^{-N_r \sum_{j=1}^{r} \tilde{m}_j}. \tag{19}
\]

To deduce coding and diversity gains from (19), we use the definitions of the diversity \( G_d \) and coding gain \( G_c \) in (4). Comparing (19) with the definition of diversity and coding gains in (4),
we see that the diversity and coding gains for the MIMO system in (3) are given by

\[ G_d = N_r \sum_{j=1}^{r} \tilde{m}_j, \]  

(20)

\[ G_c = \frac{\Omega}{4} \prod_{j=1}^{r} \left( \frac{\lambda_j}{\tilde{m}_j} \right)^{\tilde{m}_j \sum_{j=1}^{r} m_j}, \]  

(21)

Under Rayleigh fading \((m = 1)\), \(\tilde{m}_j = 1\), and we retrieve the expressions for the diversity and coding gains from (20) and (21) that were first derived in [15], [16], as follows,

\[ G_d = rN_r, \]

\[ G_c = \frac{\Omega}{4} \prod_{j=1}^{r} \lambda_j^{1/r}. \]

Assuming fading is possibly less or more severe than Rayleigh fading (i.e., \(m \neq 1\)), what are the bounds on the diversity and coding gains and how can optimum gains be achieved? In the next subsection, we address these questions and extend the design criteria in [15], [16] for Nakagami-\(m\) fading, MIMO channels.

D. Design Criteria

The purpose of this section is twofold. First, we find a property of space-time codes that allows for fully exploiting reduced fading severity to increase the diversity gain. Second, we use this property to examine why orthogonal space-time block codes (OSTBCs) are able to achieve the maximum possible diversity gain under Nakagami-\(m\) fading, while classes of non-orthogonal space-time block codes (NOSTCs) are unable to achieve this optimum diversity gain.

The fading severity under Nakagami-\(m\) fading is captured by \(m \geq 1/2\). We want to find what property of space-time codes allows for exploiting reduced fading severity to increase the diversity and coding gains. To this end, we use the bound on \(\tilde{m}_j\) in (17) to bound the diversity and coding gains in (20) and (21), respectively. Substituting bounds on \(\tilde{m}_j\) in (20) gives us that diversity gain is bounded as follows:

\[ rN_r \min(1, m) < G_d \leq rN_r \max(1, m) \quad \text{for} \quad m \geq 1/2. \]  

(22)

Similarly, substituting for the bound on \(\tilde{m}_j\) in (21), the coding gain \(G_c\) satisfies

\[ \frac{\Omega}{4} \prod_{j=1}^{r} \left( \frac{\lambda_j}{\max(1, m)} \right)^{\min(1, m)} \leq G_c \leq \frac{\Omega}{4} \prod_{j=1}^{r} \left( \frac{\lambda_j}{\min(1, m)} \right)^{\max(1, m)} \quad \text{for} \quad m \geq 1/2. \]  

(23)
We observe that the upper bounds in (22) and (23) are achieved when the respective upper bounds on $\tilde{m}_j$ are achieved. The upper bound on $\tilde{m}_j$ in (17) is achieved when $\sum_{j'=1}^{N_t} |u_{j',j}|^4 = 1$, which is only possible if, $\forall u_{j',j} \in \mathbb{C}$: $\sum_{j'=1}^{N_t} |u_{j',j}|^2 = 1$, $|u_{j',j}| = 1$ for some $j' = j_0$, and $|u_{j',j}| = 0$ otherwise. In other words, the upper bounds on the diversity and coding gains can be achieved if and only if the codeword results in a matrix $U$ such that there is only one non-zero element in each column of $U$, which is also equal to 1. Alternatively, we can consider maximizing $\tilde{m}_j$ for $1/2 \leq m < 1$ and $m \geq 1$. The two cases have a similar structure. We show that this is indeed the only case in which the upper bound on $G_d$ is achieved.

Consider the case when $m \geq 1$. From (17), the maximum value of $\tilde{m}_j$ is $m$. Also, this maximum is obtained when the denominator of (14), i.e., the expression in (15), is equal to 1. Hence, it follows from (15), that the optimal codeword results in:

$$
(1 - m) \sum_{j'=1}^{N_t} |u_{j',j}|^4 + m = 1
$$

$$
\Leftrightarrow (1 - m) \left\{ \sum_{j'=1}^{N_t} |u_{j',j}|^2 (|u_{j',j}|^2 - 1) \right\} = 0, \quad (24)
$$

where (a) follows from that the Euclidean norm of rows of $U$ is unity. For (24) to be satisfied, there are only two possible solution: $|u_{j',j}|^2 = 0$ or $|u_{j',j}|^2 = 1$. But it cannot be that all the elements of $U$ are zeros since $\sum_{i=1}^{N_t} |u_{j',j}|^2 = 1$. Hence, it must be that, in every column of $U$, there must be only one element whose value is 1. Equivalently, we must have that $\forall j' = 1, 2, \ldots, N_t$,

$$
|u_{j',j}|^2 = \delta_{j'j_0} \triangleq \begin{cases} 
1 & \text{for } j' = j_0, \\
0 & \text{otherwise}, 
\end{cases} \quad (25)
$$

where $j_0 \in \{1, 2, \ldots, N_t\}$ denotes the index of the non-zero element in each column of $U$. The identity matrix has one nonzero entry in each column that is itself equal to 1. Therefore, any space-time coding scheme that generates a matrix $U$ (cf. section III-A) that is either the identity or whose columns are permutations of those the identity matrix (of size $N_t \times N_t$) achieves the maximum possible diversity gain of $rmN_r$. We similarly obtain the result in (25) for the case when $1/2 \leq m < 1$.

We have established that space-time codes must satisfy (25) to maximize the diversity gain, thereby, improving their error performance. Now, we consider OSTBCs and classes of NOSTBCs.
to establish further design criteria.

- **OSTBCs** produce a unitary matrix \( U \) that is the identity matrix [27, pp. 145–147], which satisfies (25). This follows from the definition of OSTBCs as having a matrix \( S \) whose rows are orthogonal (i.e., the dot product of one row with the conjugate transpose of another equals zero). Let \( s_i \) denote the \( i \)th row of \( S \), and \( e_i \) denote the \( i \)th row of \( E \). Then, since \( S \) and \( E \) are orthogonal matrices with the same structure, the rows of \( S - E \) are also orthogonal, i.e., \( (s_i - e_i)(s_j - e_j)^H = 0 \), for \( i \neq j \) and \( \forall i, j = 1, 2, \ldots, N_t \). As a result, the off-diagonal entries of matrix \( (S - E)(S - E)^H \) are all zeros, and, hence, \( U \) is the identity. Therefore, OSTBCs satisfy (25) and, hence, can achieve the upper bounds on the diversity gain in (22).

- For \( 1/2 \leq m < 1 \), OSTBCs can achieve the maximum diversity gain in (22) of \( r N_r \). Hence, designers must maximize \( r \), the rank of \( (S - E)(S - E)^H \), which is also known as the **rank criterion** [15]. OSTBCs result in a matrix \( U \) that is full rank. Further, since \( r \leq \min(N_t, T) \), the length of the codeword \( T \) should be sufficiently large to fully exploit the benefits in terms of diversity gain due to having multiple antennas at the transmitter, i.e., \( T \geq N_t \).

- Similarly for \( m \geq 1 \), we maximize the diversity gain that space-time codes can achieve in (22), which is \( mr N_r \). OSTBCs produce a matrix \( U \) that is full rank, and we ensure enough block length to utilize multiple antennas at the receivers by having \( T \geq N_t \).

- To overcome excessive decoding delays, \( T \) must be chosen to be equal to \( N_t \).

In general, designers target a specific diversity gain and then optimize the coding gain given that target [15]. Maximizing the diversity gain does not necessarily maximize the coding gain. Indeed, there exists a trade-off between these two gains.

- Achieving the upper bounds on the diversity gain \( G_d \) of \( r N_r \) and \( mr N_r \) for \( 1/2 \leq m < 1 \) and \( m \geq 1 \) in (22), respectively, is possible if and only if \( \sum_{j=1}^{N_t} |u_{j',j}|^4 = 1 \). But if \( \sum_{j'=1}^{N_t} |u_{j',j}|^4 = 1 \), we obtain an equality at the lower bound of the achievable coding gain \( G_c \) in (23) (i.e., achieving the upper bound on the diversity gain results in achieving the lower bounds on the coding gain). Hence, we observe a trade-off between \( G_d \) and \( G_c \).

- First we consider the case in which the maximum possible diversity gain is required and find the achievable coding gain. For \( 1/2 \leq m < 1 \), OSTBCs that achieve the upper bound on the diversity gain of \( r N_r \) in (22) can only achieve the lower bound on the coding gain...
of $\Omega_4 \prod_{j=1}^{r} \lambda_j^m$ in (23). Hence, we find a trade-off between maximizing the diversity gain using OSTBCs and maximizing the coding gain. Nonetheless, we can maximize the coding gain of $\Omega_4 \prod_{j=1}^{r} \lambda_j^m$ given we are targeting an optimum diversity gain. This is equivalent to maximizing the product of the eigenvalues $\{\lambda_j\}_{j=1}^r$ of $(S - E)(S - E)^H$, which is also known as the determinant criterion [15].

- A similar argument follows for $m \geq 1$. OSTBCs can achieve the optimum diversity gain of $mrN_r$. But this would result in achieving the lower bound on the coding gain in (23) of $\Omega_4 \prod_{j=1}^{r} \left( \frac{\lambda_j}{m} \right)^{\frac{1}{rm}}$. Maximizing this coding gain is now equivalent to maximizing the product of the eigenvalues $\{\lambda_j\}_{j=1}^r$ of $(S - E)(S - E)^H$ (since $m$ and $r$ are fixed given a certain fading severity and a target diversity gain, respectively).

- To achieve the upper bound on the coding gain in (23), we must relax the optimality constraint on the diversity gain. Indeed, the upper bounds on the coding gains can only be achieved if and only if $\sum_{j'=1}^{N_t} |u_{j',j}|^2 \to 0$, which would also result in achieving the lower bound on the diversity gains in (22). This optimization of the coding gain can be accomplished using NOSTBCs that result in the maximum possible non-zero entries in $U$. Then, the codeword must maximize the product of the eigenvalues $\{\lambda_j\}_{j=1}^r$ of $(S - E)(S - E)^H$ to obtain the maximum coding gain. We observe that the determinant criterion applies both when achieving the maximum possible coding gain or the the maximum possible diversity gain is the objective. But the maximum achievable coding gain results when lower bound on the diversity gain is achieved.

We showed that space-time codes can achieve the maximum diversity gain of $mrN_r$ if and only if they produce a matrix $U$ that is the identity (of size $N_t \times N_t$) or whose columns are a permutation of the columns of the identity matrix. Furthermore, there is a trade-off between the diversity and coding gains.

IV. EFFECT OF BLOCKAGE ON THE AVERAGE PEP

Now that we have established design criteria for space-time codes under Nakagami-$m$ fading for systems with multiple antennas, we find it instructive to study the effect of blockage on the error performance of mm-wave systems. As a consequence of the high directivity of the mm-wave channel [2], [3], [5], [6], blockage due to humans and other objects prominently impacts the performance of mm-wave systems. In this section, we use stochastic geometry to model
obstacles in indoor environments and derive the average PEP in terms of parameters that dictate
the distribution of obstacles in a room.

A. Preliminaries

The setup we are considering is a closed room environment. We use a Cartesian coordinate
system $xyz$, where $z$ is along the height of the room. We fix the locations of the transceivers
at a distance $d$ apart. The distance $D_0$ between an obstacle and the transmitter is uniformly
distributed, i.e., $D_0 \sim U(0, d)$. We assume obstacles are cylindrically-shaped with heights $H_i$
and radii $R_i$ that are uniformly distributed within a certain range, i.e., $H_i \sim U(h_{\text{min}}, h_{\text{max}})$ and
$R_i \sim U(r_{\text{min}}, r_{\text{max}})$. The number of obstacles in a room $N$ is modeled via a PPP with density
$\lambda_b$ [number of obstacles per unit volume], i.e., $N \sim \text{Poisson}(\lambda_b)$ [5], [6], [22].

We model blockage as in [5], [6] using a discrete random variable $\Gamma \in [0, 1]$ that penalizes a
LOS communication with probability (w.p.) $P_{\text{LOS}}$ and a non-line of sight (NLOS) communication
w.p. $1 - P_{\text{LOS}}$ as:

$$\Gamma = \begin{cases} 
\gamma_1 \triangleq \left( \frac{d}{d_{\text{ref}}} \right)^{-\alpha_L} & \text{w.p. } P_{\text{LOS}}, \\
\gamma_2 \triangleq \left( \frac{d}{d_{\text{ref}}} \right)^{-\alpha_N} & \text{w.p. } 1 - P_{\text{LOS}},
\end{cases}$$

(26)

where $d$ is the distance between the transmitter and the receiver, $d_{\text{ref}}$ is a reference distance,
$\alpha_L$ and $\alpha_N$ are exponent decays experienced by the transmitted signals during LOS and NLOS
communications, respectively. Then, $\Gamma$ multiplies our channel matrix $H$; hence, our system model
is now given by

$$R = \sqrt{\rho} \Gamma HS + Z.$$  (27)

In practical communication systems, the SNR due to NLOS communication is penalized more
stringently than that due to LOS communication, i.e., $\alpha_N > \alpha_L$ [11].

B. Average PEP

The average PEP over the channel coefficients for the system in (27) is (cf. section III-C):

$$\Pr\{S \to E|\Gamma\} = \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \frac{(\bar{m}_j/\bar{\Omega})^{\bar{m}_j}}{(\rho/4)\lambda_j \Gamma + (\bar{m}_j/\bar{\Omega})^{\bar{m}_j}}.$$  (28)
Now, by averaging (28) over $\Gamma$, the average PEP can be written as:

$$\Pr\{S \rightarrow E\} = E_{\Gamma}\{\Pr\{S \rightarrow E|\Gamma\}\}$$

$$= 1 \prod_{i=1}^{N_r} \prod_{j=1}^{r} \left(\frac{\bar{m}_j}{\bar{\Omega}}\right)^{\bar{m}_j} P_{\text{LOS}} + \frac{1}{2} \prod_{i=1}^{N_r} \prod_{j=1}^{r} \left(\frac{\bar{m}_j}{\bar{\Omega}}\right)^{\bar{m}_j} (1 - P_{\text{LOS}}). \quad(29)$$

To model the effect of blockage on the diversity and coding gains, we express $P_{\text{LOS}}$ as a function of known parameters such as the density of obstacles and their dimensions and the transceivers’ locations. Also, we are interested in finding whether $P_{\text{LOS}}$ is also a function of the SNR since this would have implications on the diversity and coding gains.

**C. Expressing $P_{\text{LOS}}$ as a Function of Environmental Properties**

We want to find $P_{\text{LOS}}$ as a function of known environmental properties. To this end, we first find the effective density of obstacles. While the density of obstacles in the room is $\lambda_b$, not all obstacles present in a room can cause a blockage. In fact, only obstacles above a certain height and located between the transceivers can block the LOS communication, as shown in Fig. 2(a). Similarly, only if there exists an obstacle within a certain blocking region with a large enough
diameter can there be a blockage due to its radius, as shown in Fig. 2(b). These two events of blockages due to heights and radii of obstacles reduce original density to an effective one. More specifically, using the Thinning Theorem from stochastic geometry [22], the effective density of obstacles \( \lambda' b \) is given by:

\[
\lambda' b = \lambda b \Pr\{\xi_1\} \Pr\{\xi_2\},
\]

where \( \Pr\{\xi_1\} \) is the probability of blockage event \( \xi_1 \) due to the height of an obstacle blocking the LOS between the transceivers, and \( \Pr\{\xi_1\} \) is the probability of blockage event \( \xi_2 \) due to the radii of obstacles.

To find \( \Pr\{\xi_1\} \), consider Fig. 2(a). We fix the heights of the transceivers, as \( h_t \) and \( h_r \), and the distance between them as \( d \). From Fig. 2(a), a blockage event due to the height of obstacles occurs if \( H_0 \) is greater than \( h(D_0) \), where \( H_0 \) is the height of the tallest obstacle in the blocking region and \( D_0 \) is the distance between the tallest obstacle and the transmitter. For \( n \) obstacles in the blocking region with heights \( H_1, H_2, \ldots, H_n \), \( H_0 = \max(H_1, H_2, \ldots, H_n) \). The volume of the blocking region, which is the space in which obstacles could cause a blockage event, is \( dh_tr_{max} \). Thus, the number of obstacles in the blocking region is \( n = \lceil\lambda_d dh_tr_{max}\rceil \). Hence, as we show in more detail in Appendix A, \( \Pr\{\xi_1\} \) is given by:

\[
\Pr\{\xi_1\} = \mathbb{E}_{D_0}\left\{ \Pr\left\{ H_0 > h(D_0) | D_0 = d_0 \right\} \right\}
= 1 - \frac{1}{(r_{max} - r_{min})^n} \sum_{k=0}^{n} \binom{n}{k} (h_t - h_{min})^{n-k} \left( \frac{h_r - h_t}{k+1} \right)^k.
\]

(31)

Finding \( \Pr\{\xi_2\} \), follows a similar procedure as illustrated by Fig. 2(b). Upon fixing \( d \), only an object with a radius greater than \( |y| \) can cause a blockage event, where \( y \) is the distance of the line perpendicularly joining the center of the blocking object to the LOS between the transceivers. For \( n \) obstacles in the blocking region with radii \( R_i, i = 1, \ldots, n \), define \( R_0 = \max(R_1, R_2, \ldots, R_n) \). Then, as shown in Appendix A, \( \Pr\{\xi_1\} \) is given by:

\[
\Pr\{\xi_2\} = \mathbb{E}_Y\left\{ \Pr\left\{ R_0 > |y| | Y = y \right\} \right\}
= 1 - \frac{1}{(r_{max} - r_{min})^n} \sum_{k=0}^{n} \binom{n}{k} (-r_{min})^{n-k} \frac{r_0^k}{k+1}.
\]

(32)

Now, to evaluate the average PEP in (29), we find the probability of having a LOS communication between the transceivers \( P_{LOS} \). Let \( K' \) denote the effective number of obstacles that can
cause blockage, i.e., $K' \sim \text{Poisson}(\lambda'_b)$. Then, the probability of having a LOS communication is given by:

$$P_{\text{LOS}} = \Pr \{K' = 0\} = \exp (-\lambda'_b).$$  \hspace{1cm} (33)

We observe from (29) that $P_{\text{LOS}}$ is not a function of the SNR. And, since the diversity gain is defined in terms of the behavior of the PEP as the SNR goes to infinity, the diversity gain is not affected by the presence of obstacles. This also makes sense since the diversity gain utilizes independent channel links due to using multiple antennas at the transceivers which is independent of whether the link is a LOS or a NLOS one. By averaging the PEP, and then averaging the coding gain over $\Gamma$, we obtain

$$G^b_c = \left\{ P_{\text{LOS}} \left( \frac{d}{d_{\text{ref}}} \right)^{-\alpha_L} + (1 - P_{\text{LOS}}) \left( \frac{d}{d_{\text{ref}}} \right)^{-\alpha_N} \right\} G_c, \hspace{1cm} (34)$$

where $G^b_c$ is the effective coding gain when the effect of blockage is considered, and $G_c$ is given by (21). We observe from (34) that for $\alpha_L = \alpha_N = 0$, the SNR is not reduced due to either LOS or NLOS communication. Hence, the coding gain with blockage is maximized, i.e., $G^b_c = G_c$. As $\alpha_N$ becomes large, only the first term of the sum in (34) contributes to $G^b_c$. This case applies when the path loss for NLOS communication is very high, rendering impossible any reliable communication over the NLOS paths. In such a case, $G^b_c = P_{\text{LOS}} \gamma_1 G_c$.

V. NUMERICAL RESULTS

In this section, we illustrate our design criteria and the effect of blockage on the error performance of space-time codes. More specifically, our simulations show that any class of NOSTBCs that produces a matrix $U$ that has non-zero entries along the same column (or row) cannot fully exploit the reduced severity in Nakagami-$m$ fading to increase the diversity gain. OSTBCs, however, can fully exploit the reduced severity of Nakagami-$m$ fading to increase the diversity gain since they produce a unitary matrix $U$ that is the identity. We also examine the effect on the bit error rate (BER) of incorporating the blockage model studied in the previous section to the MIMO system, as in (27). We denote the number of transmitters $N_t$ and the time slots $T$ (i.e., channel uses) used by a codeword as $N_t \times T$.

In Fig. 3(a), we simulate under Nakagami-$m$ fading the space code (SC) designed to exploit the directivity of the mm-wave channel in [8]. We see that as the value of $m$ doubles, the SC $(2 \times 2)$ only partially exploits the reduced fading severity to achieve a higher diversity gain,
Fig. 3: BER versus SNR for $(2 \times 2)$ and $(4 \times 4)$ space-time block codes under different Nakagami-$m$ fading channel conditions and blockage considerations.

denoted by the slope of the BER curve. This increase in the diversity gain is equal to about 1 dB (where dB denotes decibels) in Fig. 3(a), i.e., doubling the value of $m$ results in an increase in the diversity gain by a factor less than 2. On the other hand, the OSTBC $(2 \times 2)$ in Fig. 3(b), which is the Alamouti scheme [37], doubles the diversity gain every time we double the value of $m$. Specifically, for SNRs higher than or equal to 6 dB we observe that $\Delta G_d = 2$ dB, $G_d(m = 2) - G_d(m = 1) \approx G_d(m = 4) - G_d(m = 2) \approx 3$ dB (diversity gain is doubled when $m$ is doubled), where $\approx$ designates both sides of the inequality are with 0.3 dB, $G_d(m) = 10 \log_{10} G_d$, $G_c(m) = 10 \log_{10} G_c$, and $G_d(m)$ and $G_c(m)$ are computed from BER versus SNR curves as in [27, pp. 11–12]. Hence, this demonstrates that the OSTBC $(2 \times 2)$ fully exploits reductions in the fading severity to increase the diversity gain. The reason why orthogonal codes are able to achieve the maximum diversity order of $m r N_r$ is that they produce a unitary matrix $U$ that is the identity, resulting in the expression for $\tilde{m}_j$ being equal to its maximum value of $m$ (cf. section III-D).
Figure 3(c) shows a simulation of an OSTBC \((2 \times 2)\) under the system model in (27) which considers the effect of blockage on MIMO systems. We simulate a typical, indoor wireless communications scenario using equations (26) – (33). The scenario uses values for \(\alpha_L\) and \(\alpha_N\) in (26) found by the experimental study of the indoor wireless channel at 60 GHz in [11], and \(\gamma_1 = 0.9\) and \(\gamma_2 = 0.5\) in (26). Varying density \(\lambda_b\), the distances between transceivers, and the heights of transceivers changes \(P_{\text{LOS}}\) as given by (30) – (33). We observe from Fig. 3(c) that increasing the probability of a LOS communication does not affect the slope of the BER versus SNR curve but shifts the curve to the left. That is, the coding gain which is represented by translations of the BER curve along the SNR axis is affected by blockage, while the diversity gain which represented by the slope is not. With regards to the error performance, we find that blockage does not influence the diversity order (the slope of the BER curve over SNR), as indicated by the constant slopes of the BER curves in Fig. 3(c). The coding gain increases by increasing \(P_{\text{LOS}}\) as shown by the shift in the BER curve to the left. A BER of \(10^{-3}\) is achieved at an SNR of 7 dB for \(P_{\text{LOS}} = 0.9\). Reducing \(P_{\text{LOS}}\) to 0.5 and 0.1 results in a shift of the BER curve by about 2.8 dB and 3.2 dB to the right, respectively, reducing the coding gain (i.e., \(\Delta G^{\text{dB}}_c < 3.2\) dB but \(\Delta G^{\text{dB}}_d \approx 0\) dB). These shifts in the BER curve and the constancy of its slopes as \(P_{\text{LOS}}\) changes are consistent with our analysis in section IV that the coding gain is affected by blockage and the diversity gain is not.

We repeat our simulation for a higher number of transmit and receive antennas as shown in Figures 3(d) – 3(f). Specifically, we simulated a simple RC \((4 \times 4)\) with four transmit and receive antennas and the OSTBC \((4 \times 4)\) proposed in [16, eq. (4)], i.e., the codeword \(D_{4 \times 4}\) in [27, eq. (7.51)]. Figure 3(d) shows that even for a larger number of transmit antennas, a simple RC is unable to improve the diversity gain as we reduce the severity of fading. On the other hand, as shown in Fig. 3(e), the OSTBC \((4 \times 4)\) is able to approximately double the diversity gain as we double the value of \(m\). A BER of \(10^{-3}\) is achieved at 3.5 dB for a \(P_{\text{LOS}}\) of 0.9. As \(P_{\text{LOS}}\) decreases to 0.5 and 0.1, the BER of \(10^{-3}\) is achieved at about 5 dB and 6 dB, respectively. But, changing \(P_{\text{LOS}}\) does not affect the slopes of BER curves. Thus, the coding gain is affected by blockage but the diversity gain is not. Hence, our design criteria are illustrated to hold for a larger number of transmit antennas as well. Furthermore, increasing \(P_{\text{LOS}}\) results only in a shift in the BER curve to the left, increasing the coding gain, and does not affect the diversity gain (slope of the BER curve), as shown in Fig. 3(f).
Note that antenna arrays with a large number of antennas at mm-wave frequencies can be used to electronically steer main-lobes. But, the benefits in terms of diversity and coding gains are only achieved due to signals traveling through independent paths via multiple antennas. For instance, utilizing two main antenna arrays at the transmitter and receiver such that each has an independent main-lobe corresponds to the simulations for the $2 \times 2$ system provided. Hence, when a large number of antennas is used in antenna arrays, it is important to consider the number of independent main-lobes when designing for particular coding and diversity gains rather than the number of antennas in each antenna array.

VI. CONCLUSION

We derived an upper bound on the PEP assuming the channel coefficients modeling small-scale fading are complex random variables with Nakagami-\emph{m}-distributed amplitudes and uniformly distributed phases. The maximum diversity gain of $mrN_t$ is achievable by space-time block codes that result in a unitary matrix $U$ whose columns are permutations of those of the identity matrix. Otherwise, space-time codes that produce a matrix $U$ that has at least two non-zero entries along the same column (or row) achieve a diversity gain strictly less than $mrN_t$. Furthermore, we showed that there is a trade-off between the diversity and coding gains. We also studied the effect of blockage due to humans and other objects on the PEP. Our analysis and simulations show that blockage results only in a shift of the BER versus SNR curve (i.e., affects coding gain), and does not affect the slope of the BER curve (i.e., does not affect the diversity gain). We expressed the reduction in the coding gain as a function of the probability of a LOS communication, power exponents, the distance between transceivers, and the density of obstacles.

As a future research direction, design of space-time codes that satisfy the design criteria proposed is an an appealing research problem. Furthermore, the PEP and design criteria for space-time codes could be analyzed for Gamma and log-normal fading channels. Also, analyzing the average error performance of space-time codes used in mm-wave cellular networks is an interesting point to investigate.

APPENDIX A

EVALUATING THE PROBABILITY OF BLOCKAGE DUE TO HEIGHTS AND RADII OF OBSTACLES

We find the expressions for $Pr\{\xi_1\}$ and $Pr\{\xi_1\}$, which are the probabilities of blockage events due to the heights and radii of obstacles, respectively. From Fig. 2(a), a blockage event due to
the height of an obstacles occurs if \( H_0 \) is greater than \( h(D_0) \), where \( H_0 \) is the maximum height of the obstacles in the blocking region, i.e., for \( n \) obstacles in the blocking region with heights \( H_1, H_2, \ldots, H_n \), \( H_0 = \max \{ H_1, H_2, \ldots, H_n \} \). Hence, the probability of blockage due to heights of obstacles is given by:

\[
\Pr \{ \xi_1 \} = \Pr \left\{ h_0 > h(D_0) \right\}
\]

\[
= 1 - \mathbb{E}_{D_0} \left\{ F_{H_0|D_0=d_0} \left( \frac{h_t - h_r}{D_0} D_0 + h_t \right) \right\}
\]

\[
= \begin{cases} 
0 & h(D_0) \geq h_{\text{max}} \\
\mathbb{E}_{D_0} \left\{ \frac{(h_t - h_r)}{d_0} D_0 + h_t - h_{\text{min}} \right\}^n & h_{\text{min}} \leq h(D_0) < h_{\text{max}} \\
1 & h(D_0) < h_{\text{min}}.
\end{cases}
\]

Now, evaluating (31) for \( h_{\text{min}} \leq h(D_0) < h_{\text{max}} \), using the fact that \( D_0 \sim U(0, d) \), and using the binomial theorem, we obtain:

\[
\Pr \{ \xi_1 \} = 1 - \int_0^d \left\{ \frac{(h_t - h_r)}{d_0} \frac{D_0 + h_t - h_{\text{min}}}{h_{\text{max}} - h_{\text{min}}} \right\}^n dd_0
\]

\[
= 1 - \frac{1}{(h_{\text{max}} - h_{\text{min}})^n} \sum_{k=0}^n \binom{n}{k} (h_t - h_{\text{min}})^{n-k} \left( \frac{h_t - h_r}{k+1} \right)^k,
\]

where \( n \) is the number of obstacles in the blocking region and is equal to \( \lceil \lambda_b d h_t r_{\text{max}} \rceil \) and (a) follows from the binomial expansion of the integrand.

Finding \( \Pr \{ \xi_2 \} \) follows a similar procedure. From Fig. 2(b), the starting point of the proof is \( \Pr \{ \xi_2 \} = \Pr \{ R_0 > |y| \} \), where \( R_0 \) is the radius of the widest obstacle in the blocking region, i.e., \( R_0 = \max \{ R_1, R_2, \ldots, R_n \} \). The rest follows as in the previous derivation to obtain

\[
\Pr \{ \xi_2 \} = \mathbb{E}_Y \left\{ \Pr \left\{ R_0 > |y| \middle| Y = y \right\} \right\}
\]

\[
= 1 - \frac{1}{(r_{\text{max}} - r_{\text{min}})^n} \sum_{k=0}^n \binom{n}{k} (-r_{\text{min}})^{n-k} \frac{r^k}{k+1}.
\]

**References**


